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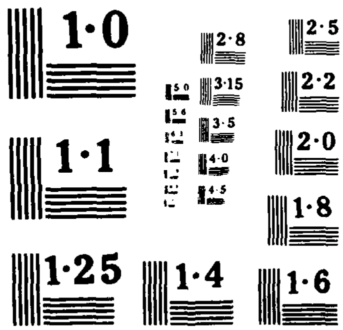
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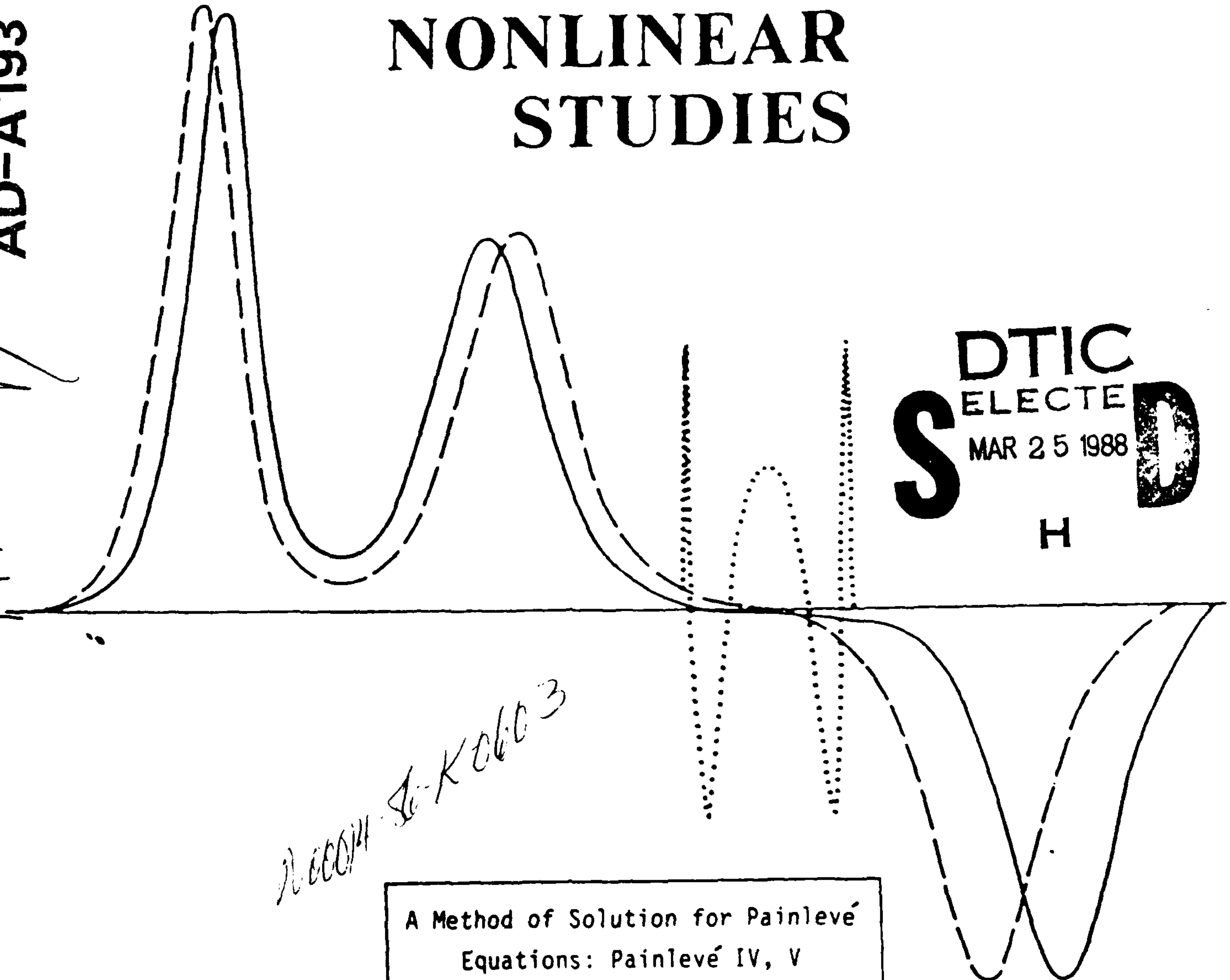



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A Method of Solution for Painlevé Equations: Painlevé IV, V

by

A.S. Fokas, U. Mugan, M.J. Ablowitz

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# A Method of Solution for Painlevé Equations: Painlevé IV, V

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## 1. INTRODUCTION.

The mathematical and physical significance of the six Painlevé transcendents, PI-PVI, has been well established. Their mathematical importance originates from the following: i) P. Painlevé [1] and B. Gambier [2], at the turn of the century, classified all equations of the form $q_{tt} = F(q_t, q, t)$ where F is rational in $q_t$, algebraic in q and locally analytic in t, which have the Painlevé property, i.e. their solutions are free from movable critical points [3]. Within a Möbius transformation, they found fifty such equations; these equations can either be integrated in terms of known functions or can be reduced to one of the six Painlevé transcendents. ii) R. Fuchs [4] and R. Garnier [5] considered Painlevé equations as the isomonodromic conditions for suitable linear systems with rational coefficients possessing regular singular points. In other words, Fuchs and Garnier established the existence of compatible linear systems which turn out to be the analogue for Painlevé equations of the so called Lax pairs [6] for solvable nonlinear evolution equations. The condition of isospectrality is now replaced by isomonodromicity. However, apparently the above authors did not pose the question of using these Lax pairs to integrate the Painlevé transcendents. iii) Ablowitz, Ramani and Segur

[7] discovered a remarkable connection between equations with the Painlevé property and nonlinear PDE's solvable by the inverse scattering transform (IST). Invariant solutions of such PDE's satisfy equations with the Painlevé property. (More precisely, associated with such a PDE consider an ODE which describes those solutions of the PDE which remain invariant under the action of some Lie-point group; then all solutions of this ODE which can be obtained via the IST of the corresponding PDE have the Painlevé property, see also [8]). For example, proper exact reductions of the Korteweg-deVries (KdV) equation lead to PI and PII [9]; PII and special cases of PIII and PIV can be obtained from the exact similarity reduction of the modified KdV, sine-Gordon and the nonlinear Schrödinger equations, respectively [10]; special cases of PVI can be obtained from exact reductions of the three-wave resonant interactions [11] and from the Ernst [12] equation [13]. iv) H. Flaschka and A. Newell [14], M. Jimbo, T. Miwa and K. Ueno [15] considered Painlevé equations as isomonodromic conditions for suitable linear systems possessing both regular and irregular singular points. These systems appear more suitable than the linear systems introduced in [4] and [5] for both integrating the Painlevé transcendents, as well as for studying their asymptotic behavior [16]. Garnier [17] also considered linear equations with irregular singular points in connection with the Painlevé equations but apparently did not pose the question of using this connection to integrate these equations.

The physical significance of the Painlevé transcendents follows from their applicability to a wide range of important physical problems.

Painlevé equations appear in nonlinear waves (see iii) above) in quantum field theory, in statistical mechanics [18]-[30], etc.

Associated with the exact integrability of Painlevé equations there exist two interrelated aspects: (i) Find a method for generating particular solutions. (ii) Find a method for solving the initial value problem. With respect to (i) we note: ($\alpha$) There exist certain explicit transformations which map solutions of a given Painlevé equation to solutions of the same equation but with different values of the parameters. Such maps, called Schlesinger transformations, were given by Lukashevich and Gromak [31]-[34] for PII-PV and by Fokas and Yortsos [35] for PVI. Furthermore it turns out that for certain choices of their parameters PII-PVI admit rational solutions as well as one-parameter families of solutions expressible in terms of Airy [2], [36], Bessel [37], Weber-Hermite [38], Whittaker [39] and hypergeometric [40] functions respectively. Using the above transformations and special solutions, one can construct (for certain choices of the parameters) various elementary solutions of PII-PVI. These solutions are either rational or are functions which are related, through repeated differentiations and multiplications, to the above mentioned classical transcendental functions. ($\beta$) Ablowitz and Segur [10] characterized a non-elementary one parameter family of solutions of PII through a Gel'fand-Levitan-Marchenko integral equation of the Fredholm type. ($\gamma$) Fokas and Ablowitz [41] characterized a two-parameter family of solutions of PII using a matrix system of Fredholm integral equations. However, in both ($\beta$) and ($\gamma$) the free parameters were not related to the initial data of PII.

The main focus of this paper is to give a method to solve the initial value problem of the Painlevé equations. In this respect we note: (i) Flaschka, Newell, Jimbo, Miwa and Ueno introduced a new powerful approach for studying the associated initial value problem; solving such an initial value problem is essentially equivalent to solving an inverse problem for a certain isomonodromic linear equation (see (iv) above). In analogy with the IST method introduced by Gardner, Greene, Kruskal and Miura [42] we call the above method an inverse monodromic transform method (IMT). (ii) Flaschka and Newell [14] applied the above method to the solution of PII and to a special case of PIII. They formulated the inverse problem in terms of what the authors of [14] call a system of singular integral equations. (iii) Jimbo, Miwa and Ueno [15] considered the Painlevé equations within the larger program of study of monodromy preserving deformations for a first order matrix system of ODE's having regular or irregular singularities of arbitrary rank. The inverse problem is solved in terms of formal infinite series uniquely determined in terms of certain monodromy data. (iv) Fokas and Ablowitz demonstrated that the inverse problem of PII can be formulated as a matrix, singular, discontinuous, homogeneous Riemann-Hilbert (RH) problem defined on a complicated contour. This has conceptual and practical implications: Conceptually, it becomes clear that there is a unified approach to solving certain initial value problems for equations in 1, 1+1 (one spatial and one temporal) and 2+1 dimensions. Using techniques from RH theory, the RH problem can be simplified substantially (it can be mapped to a series of regular, continuous RH problems, each defined on the real axis).

In this paper we present a general method for solving the initial value problem associated with a given Painlevé equation. This method, which simplifies and extends ideas of [43], involves three main steps: 1. Use classical theory of linear ODE's to formulate a RH problem for a function called $\Psi(z,t)$ which solves the underlying isomonodromic linear equation. This basic RH problem is, in general, a matrix singular, discontinuous, problem formulated on a complicated contour (several intersecting rays) and is uniquely defined in terms of certain monodromy data. 2. Choose the parameters of the given Painlevé equations in such a way that the above problem is nonsingular; then map the basic RH problem to series of RH problems defined on simple contours. All of these problems except one are continuous. Furthermore, some of them can be solved in closed form (in terms of a quadrature). Use certain auxiliary functions to map the discontinuous RH problem to a continuous one. Then apply the rigorous results of the RH theory, e.g. [44] to establish the existence and uniqueness of the solutions of the above continuous RH problems. 3. Use the basic RH problem to obtain Schlesinger [51], [13], transformations, shifting by an integer or by a half integer all the parameters of the given Painlevé equation. Hence, using these transformations, the study of the singular RH problem reduces to the study of the regular one. We note that for special choices of the monodromy data the basic problem can be solved in closed form. This yields particular solutions of Painlevé equations expressible in terms of the classical transcendental functions mentioned above.

The above method is applied to the solution of the initial value problem of PIV and PV: The RH problem corresponding to each of these

equations is mapped to two RH problems on simple contours, one of which can be solved in closed form, while the other can be made continuous. Furthermore, Schlesinger transformations are derived for both PIV and PV. For special choices of the monodromy data the basic RH problems for PIV and PV can be solved in closed form; this yields solutions of PIV and PV in terms of Weber-Hermite and Whittaker functions respectively.

PII is considered in [45], using a different isomonodromic spectral problem than the one used in [43]. These results, and PIII (which is related to a special case of PV [9]) will be presented elsewhere. PVI has been solved by C. Cosgrove [13] and PI remains open.

The Hamiltonian structure of the Painlevé equation is studied in [14] and [56].

## 2. THE GENERAL FRAMEWORK

### 2.1. RH Problems.

Let C be a simple, smooth, closed (or infinite) contour dividing the complex z-plane into two regions $D^+$ and $D^-$ (the positive direction of C will be taken as that for which $D^+$ is on the left).

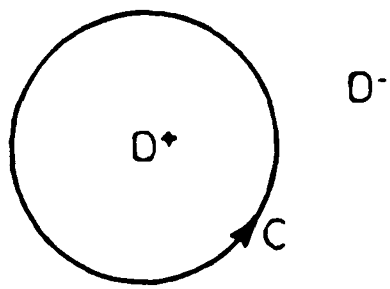


Figure 2.1

A function $\Phi(z)$ defined in the entire plane, except for points on C which will be called sectionally holomorphic if: i) the function $\Phi(z)$

is holomorphic in each of regions $D^+$ and $D^-$ except, perhaps, at $z = \infty$; ii) the function $\Phi(z)$ is sectionally continuous with respect to C, approaching the definite limiting values $\Phi^+(\zeta)$, $\Phi^-(\zeta)$ as $z$ approaches a point $\zeta$ on C from $D^+$, or $D^-$, respectively. The classical homogeneous RH problem is defined as follows [46]. Given a contour C, and a function $G(\zeta)$ which is Hölder on C and det $G(t) \neq 0$ on C, find a sectionally holomorphic function $\Phi(z)$, with finite degree at $\infty$, such that

$$\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta), \qquad \text{on} \quad C, \tag{2.1}$$

where $\Phi^{\pm}(\zeta)$ are the boundary values of $\Phi(z)$ on C. If $G(\zeta)$ is scalar, (2.1) is solvable in terms of quadratures. If $G(\zeta)$ is a matrix valued function, then (2.1) is in general solvable in terms of a system of Fredholm integral equations. Various generalizations of the above RH problem are possible. For example: i) The contour C may be replaced by a union of intersecting contours. ii) $G(\zeta)$ may have simple discontinuities at a finite number of points; in this case one allows $\Phi(z)$ to have integrable singularities in the neighborhood of these points. iii) RH problems may be considered in other than Hölder spaces (e.g. [47]): iv) One may consider inhomogeneous RH problems $\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta) + F(\zeta)$ on C.

It is interesting that the first RH problem was formulated in connection with an inverse problem (see [43] for references). Actually, RH problems are intimately related to the solution of inverse problems in 1+1 (one spatial and one temporal), 2+1 and 1 dimensions:

## 2.2. Inverse Problems in 1+1

We recall that a necessary condition for a given nonlinear equation for $q(x,t)$ to be solvable via IST is that this equation is the compati-

bility condition of a pair of linear equations. Let us consider the modified KdV

$$q_t + q_{xxx} - 6q^2 q_x = 0 \tag{2.2}$$

as an illustrative example [48]. Equation (2.2) is the compatibility condition of

$$\Psi_x(\zeta,x,t) = \zeta \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \Psi(\zeta,x,t) + \begin{pmatrix} 0 & q(x,t) \\ q(x,t) & 0 \end{pmatrix} \Psi(\zeta,x,t), \tag{2.3a}$$

$$\Psi_t(\zeta,x,t) = \begin{pmatrix} -4i\zeta^3-2iq^2\zeta & 4q\zeta^2+2iq_x\zeta+2q^3-q_{xx} \\ 4q\zeta^2-2iq_x\zeta-2q^3-q_{xx} & 4i\zeta^3+2iq^2\zeta \end{pmatrix} \Psi(\zeta,x,t). \tag{2.3b}$$

We first note that the above Lax pair is <u>isospectral</u>, i.e. $d\zeta/dt = 0$. Also it turns out that equation (2.3a) is of primary importance; equation (2.3b) plays only an auxiliary role. To solve the initial value problem, for initial data decaying as $|x| \to \infty$ , one first formulates an inverse problem for $\Psi(z,x,t)$: <u>Given appropriate scattering data reconstruct $\Psi$.</u> By studying the analytic properties of $\Psi$ with respect to z, where $\Psi$ satisfies (2.3a), one establishes that there exists a $\Psi$ which is a sectionally meromorphic function of z, with a jump along the Re z axis. This jump, as well as the residues of the poles, are given in terms of appropriate scattering data. <u>Thus the inverse problem is equivalent to a matrix, regular, continuous, RH problem defined along the Re z axis and uniquely specified in terms of the scattering data</u>.

Since in the above discussion we have only used (2.3a), it is evident that one may pose an inverse problem for an appropriate function q(x). How-

ever, this result is useful for solving the initial value problem of $q(x,t)$ only if $q$ evolves in such a way in $t$, that the scattering data is known for all $t$. If $\Psi$ evolves in $t$ according to (2.3b) (i.e. if $q$ solves (2.2)) then it turns out that the evolution of the scattering data with respect to $t$ is simple. Hence the above RH problem is specified in terms of initial scattering data; its solution yields $\Psi(\zeta,x,t)$ and then (2.3a) gives $q(x,t)$.

## 2.3. Inverse Problems in 2+1.

Let us consider the Davey-Stewartson equation (a two dimensional analogue of the nonlinear Schrödinger equation)

$$iQ_t + \frac{1}{2}(\sigma Q_{xx}+Q_{yy}) = -\lambda|Q|^2Q+\Phi Q, \quad \Phi_{xx}-\sigma^2\Phi_{yy}=2\lambda(|Q|^2)_{xx}; \quad \lambda = \pm 1, \ \sigma = \pm 1 \tag{2.4}$$

as an illustrative example [49]. A Lax pair for (2.4) is given by

$$\Psi_x = i\zeta(J\Psi-\Psi J) + q\Psi + \sigma J\Psi_y, \quad J \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q \doteq \begin{pmatrix} 0 & Q \\ \sigma^2\lambda\bar{Q} & 0 \end{pmatrix} \tag{2.5a}$$

$$\Psi_t = A_3\Psi_{yy} + A_2\Psi_y + A_1\Psi - \zeta^2(A_3\Psi-\Psi A_{30}) + 2i\zeta A_3\Psi_y + i\zeta A_2\Psi, \tag{2.5b}$$

where $A_1$, $A_2$, $A_3$, $A_{30}$ are appropriate matrix functions of $Q$, $\bar{Q}$ ( $\bar{Q}$ denotes the complex conjugate of $Q$).

The situation is conceptually similar to the case of 1+1: To solve the initial vlaue problem for $q(x,y,t)$ one first formulates an inverse problem for $\Psi(z,x,y,t)$. Depending on the value of $\sigma$ there exist two different cases (for brevity of presentation we assume non-existence of poles, i.e. non-existence of lumps): (i) $\sigma = -1$. There exists a $\Psi$ which

is a sectionally holomorphic function of $z$ and which has a jump along the Re $z$ axis. This jump is also given in terms of scattering data but it depends on them in a non-local way. Thus the inverse problem is equivalent to a non-local, matrix, regular, continuous RH problem defined along the Re $z$ axis and uniquely specified in terms of scattering data.

(ii) $\sigma = i$. There exists a $\Psi$ which is bounded for all complex $z$, but which is anlaytic nowhere in the complex $z$ plane. However, its departure from holomorphicity $\partial\Psi/\partial\bar{z}$ can be expressed linearly in terms of $\Psi$ and appropriate inverse data. Thus, now the inverse problem is equivalent to a $\bar{\partial}$ (DBAR) problem: Given $\partial\Psi/\partial\bar{z}$ reconstruct $\Psi$. The $\bar{\partial}$ problem is a generalization of a RH problem and has been studied extensively in the mathematical literature [50].

Using (2.5b), again one shows that the inverse scattering and the inverse data evolve simply in time. Hence, the above RH and $\bar{\partial}$ problems are specified in terms of initial data; their solutions yield $\Psi(\zeta,x,t,t)$ and then (2.5a) gives $q(x,y,t)$.

## 2.4. Inverse Problems in 0+1.

The Lax pair associated with the PIV equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dv}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 + \alpha)y + \frac{\beta}{y}, \tag{2.6}$$

is given by [15]

$$Y_z(z) = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} t & u \\ \frac{2}{u}(v-\theta_0-\theta_\infty) & -t \end{pmatrix} + \begin{pmatrix} \theta_0 - v & -\frac{uy}{2} \\ \frac{2v}{uy}(v-2\theta_0) & -(\theta_0 - v) \end{pmatrix}\frac{1}{z}\right] Y(z), \tag{2.7a}$$

$$Y_t(z) = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} 0 & u \\ \frac{2}{u}(v-\theta_0-\theta_\infty) & 0 \end{pmatrix}\right] Y(z). \quad (2.7b)$$

Indeed $Y_{xt} = Y_{tx}$ implies

$$\frac{dy}{dt} = -4v + y^2 + 2ty + 4\theta_0, \quad \frac{du}{dt} = -u(y+2t),$$
$$\frac{dv}{dt} = -\frac{2}{y}v^2 + \left(\frac{4\theta_0}{y} - y\right)v + (\theta_0 + \theta_\infty)y, \quad (2.8)$$

where,

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2 . \quad (2.9)$$

As in the cases of 1+1 and 2+1, solving the initial value problem of PIV reduces to solving an inverse problem for Y: <u>Reconstruct Y(z,t) in terms of appropriate monodromy data</u>. Again this inverse problem will be solved in terms of a RH problem. Thus it is essential to study the analytic properties of Y with respect to z. However, in contrast to the analogous IST problem in 1+1 and 2+1, the task here is straightforward: Equation (2.7a) is a linear ODE in z, therefore its analytic structure is completely determined by its singular points. In this particular case $z = 0$ is a regular singular point and $z = \infty$ is an irregular singular point of rank 2. Complete information about the singular point $z = 0$ is provided by the monodromy matrix $M_0$. Complete information about $z = \infty$ is provided by the monodromy matrix $M_\infty$ and by the Stokes multipliers a, b, c, d. Solutions of (2.7a), $Y_0$ and $Y_1$, normalized at zero and infinity respectively are related via a connection matrix $E_0$ with entries $\alpha_0$, $\beta_0$, $\gamma_0$, $\delta_0$. Taking into consideration the above singularities, there exist a sectionally holomorphic function Y, with jumps across four

rays, $\arg z = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{-3\pi}{4}$ and with singularities at $z = 0$, $z = \infty$. The jumps are specified by a, b, c, d and the nature of singularities by $M_0$, $M_\infty$. This leads to a matrix, singular, discontinuous RH problem, defined on the above rays and specified in terms of the monodromy data

$$\text{Monodromy Data (MD)} = \{a, b, c, d, \alpha_0, \beta_0, \gamma_0, \delta_0\}. \qquad (2.10)$$

Consistency of the above RH problem yields

$$\left(\prod_{j=1}^{4} G_j\right) M_\infty = E_0^{-1} M_0^{-1} E_0, \qquad (2.11)$$

where $G_j$ are the Stoke matrices uniquely defined in terms of the Stokes multipliers. Using (2.11) and certain similarity arguments it can be shown that all MD can be expressed in terms of two of them. Furthermore, equation (2.7b) implies that the MD are time-invariant. Hence the above basic RH is specified in terms of two initial parameters (these two initial parameters are obtained from the two initial data of PIV). The solution of this RH problem yields $Y(z,t)$ and hence (2.7a) yields $y(t)$.

This RH problem can be simplified considerably: (i) Assume $0 \le \theta_0 < 1$, $0 \le \theta_\infty < 1$, $\theta_0 \ne \frac{1}{2}$; then the above RH problem is non-singular. It is interesting that the basic RH problem can be used to obtain Schlesinger transformations which shift $\theta_0$ and $\theta_\infty$ by a half-integer. By using these transformations the general case is reduced to the regular case. (ii) The basic RH problem can be mapped to a sequence of two RH problems, one on the line $\arg z = \frac{\pi}{4}$ and the other on the line $\arg z = -\frac{\pi}{4}$. The first one is continuous (both at $z = 0$ and $z = \infty$); furthermore it can be solved in closed form. The second one is discontinuous both at

$z = 0$ and $z = \infty$. By using standard auxiliary functions [46] this discontinuous problem is mapped to a continuous one. The theory of continuous RH problems on simple contours can then be used to establish uniqueness and existence of solutions. Elementary solutions of PIV, expressible in terms of Weber-Hermite functions are obtained naturally within the above formalism.

We hope that the above discussion elucidates the connection between IST (inverse scattering transform) and IMT (inverse monodromic transform): There exists a unified approach to initial value problems in 1, 1+1, and 2+1 dimensions: Solving the initial value problem of an integrable equation: $q(t)$ or $q(x,t)$ or $q(x,y,t)$ is equivalent to solving an inverse problem for a suitable eigenfunction $\Psi(z;t)$ or $\Psi(z;x,t)$ or $\Psi(z;x,y,t)$. The inverse problem generically takes the form of a RH problem for equations in 1, 1+1, and in general the form of a $\bar{\partial}$ (DBAR) problem for equations in 2+1 (the DBAR problem being a generalization of a RH problem). To define the relevant RH or DBAR problems one needs to study the analyticity properties of $\Psi$ with respect to $z$. Furthermore these problems are uniquely defined in terms of certain asymptotic data of the underlying linear system satisfied by $\Psi$ (monodromy data in the case of equations in 1 dimension and scattering data in the case of equations in 1+1 and 2+1).

We note that the linear limit of the IST yields the Fourier transform of $q(x,t)$. In that sense IST is the nonlinear analogue of the Fourier transform [52]. Since the linear limit of the IMT is the Laplace's method for linear ODE's, the IMT is the nonlinear analogue of the Laplace's method.

## 2.5. Solution of a Matrix Continuous RH Problem.

From the above discussion it follows that solving the initial value problem of a Painlevé equation reduces to solving a matrix continuous RH problem along a simple contour. We recall that the solution of the RH problem (2.1), where $G(\zeta)$ is Hölder (i.e. all its entries satisfy $|G_{jk}(\zeta_1)-G_{jk}(\zeta_2)| < A|\zeta_1-\zeta_2|^\lambda$, for some constants A and $0 < \lambda \leq 1$, for all $\zeta$ on C) and det $G(\zeta) \neq 0$ on C is given by

$$\Phi^-(\zeta) - \frac{1}{2\pi i}\int_C \frac{d\hat{\zeta}[G^{-1}(\zeta)G(\hat{\zeta})-I]\Phi(\hat{\zeta})}{\hat{\zeta}-\zeta} = \Phi_\infty^-, \tag{2.12}$$

where $\Phi_\infty^-$ is the value of $\Phi(z)$ at infinity (we assume that $\Phi$ has a finite degree at infinity). Equation (2.12) can be obtained by writing the conditions that $\Phi^+$, $\Phi^-$ are + and - functions respectively and then replacing $\Phi^+$ by $G\Phi^-$.

In what follows we define RH problems on suitable rays. These rays are naturally defined for a given problem (see §3.2).

## 3. PAINLEVÉ IV

In this section we consider the fourth Painlevé equation (2.6). We first use equations (2.7) to study the analytic properties of $Y(z,t)$, as well as the properties of the monodromy data.

### 3.1. The Direct Problem.

**Proposition 3.1.**

Let $Y_0$ be the solution of (2.7a) analytic in the neighborhood of $z = 0$ and normalized by the requirements that $\det Y_0 = 1$ and that $Y_0$ also solves (2.7b). Let $Y_j$, $j = 1,\dots,4$ be solutions of (2.7a) analytic in the neighborhood of infinity such that $\det Y_j = 1$ and $Y_j \sim Y_\infty$ as $|x| \to \infty$ in $S_j$, $Y_\infty$ is the formal solution matrix of (2.7a) in the neighborhood of infinity, and the sectors $S_j$ are given by

$$S_1 : -\frac{\pi}{4} \le \arg z < \frac{\pi}{4}, \quad S_2 : \frac{\pi}{4} \le \arg z < \frac{3\pi}{4},$$

$$S_3 : \frac{3\pi}{4} \le \arg z < \frac{5\pi}{4}, \quad S_4 : \frac{5\pi}{4} \le \arg z < \frac{7\pi}{4}. \tag{3.1}$$

The rays $C_1,\dots,C_4$ are defined by $\arg z = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ respectively.

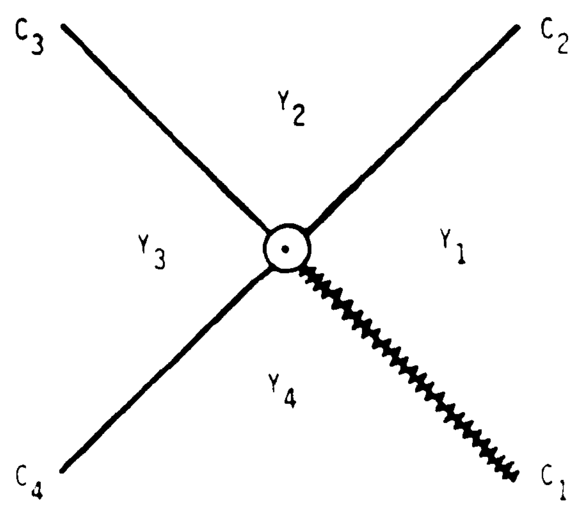


Figure 3.1

Then the analytic functions $Y_0, Y_1, \ldots, Y_4$ satisfy:

(i) $Y_0(z) \sim \hat{Y}_0(z) z^{D_0}$ as $z \to 0$; $D_0 \doteq \mathrm{Diag}(\theta_0, -\theta_0)$, $\theta_0 \neq \frac{n}{2}$, $n \in \mathbb{Z}$,

(3.2)

where $\hat{Y}_0(z)$ is holomorphic at $z = 0$. (If $\theta_0 = n/2$, $Y_0(z)$ has a logarithmic singularity).

(ii) $Y_j(z) \sim \hat{Y}_\infty(z) e^{Q(z)} \left(\frac{1}{z}\right)^{D_\infty}$ as $|z| \to \infty$, $z$ in $S_j$, $D_\infty \doteq \mathrm{Diag}(\theta_\infty, -\theta_\infty)$,

(3.3)

$Q(z) \doteq \mathrm{Diag}(q, -q)$, $q(z,t) \doteq \frac{z^2}{2} + zt$, $\hat{Y}_\infty(z)$ is holomorphic at $z = \infty$, and $\hat{Y}_\infty(z) \sim I + O(\frac{1}{z})$.

(iii) $$Y_0(ze^{2i\pi}) = Y_0(z) M_0, \quad M_0 \doteq \begin{pmatrix} e^{2i\pi\theta_0} & 2i\pi J e^{2i\pi\theta_0} \\ 0 & e^{-2i\pi\theta_0} \end{pmatrix}, \tag{3.4}$$

$J = 0$ if $\theta_0 \neq \frac{n}{2}$, $J = 1$ if $\theta_0 = \frac{n}{2}$.

(iv) $Y_2(z) = Y_1(z) G_1$, $Y_3(z) = Y_2(z) G_2$, $Y_4(z) = Y_3(z) G_3$,

$$Y_1(z) = Y_4(ze^{2i\pi}) G_4 M_\infty, \tag{3.5}$$

where

$$G_1 \doteq \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 \doteq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 \doteq \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$G_4 \doteq \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad M_\infty \doteq e^{2i\pi D_\infty}. \tag{3.6}$$

(v) $$Y_1(z) = Y_0(z) E_0, \quad E_0 \doteq \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}, \quad \det E_0 = 1. \tag{3.7}$$

Furthermore, the parameters

$$MD \doteq \{a,b,c,d,\alpha_0,\beta_0,\gamma_0,\delta_0\} \tag{3.8}$$

satisfy the following consistency condition.

(vi)
$$\left(\prod_{j=1}^{4} G_j\right)M_\infty = E_0^{-1}M_0^{-1}E_0. \tag{3.9}$$

Proof.

1. Analysis near $z = 0$:

It is well known (see for example [53]) that if the coefficient matrix of a linea differential equation has an isolated singularity at $z = 0$, the solution of the differential equation will in general be singular at $z = 0$. This solution can be obtained in the form of a formal power series; this series actually is convergent in an appropriate circle of the complex plane. In this particular case if $Y_0 = (Y_0^{(1)}, Y_0^{(2)})$ we find:

$$Y_0^{(1)}(z) = z^{\theta_0}e^{-\sigma(t)}\left\{\begin{pmatrix} 1 \\ -\frac{2v}{uy}\end{pmatrix} + \frac{1}{2\theta_0+1}\begin{pmatrix} H_{(1)}^{(1)} \\ H_{(2)}^{(1)}\end{pmatrix} z + \dots\right\}, \quad \theta_0 \neq \frac{n}{2} \tag{3.10a}$$

$$Y_0^{(2)}(z) = z^{-\theta_0}\left(1 - \frac{v}{2\theta_0}\right)e^{\sigma(t)}\left\{\begin{pmatrix} \frac{uy}{2(2\theta_0-v)} \\ 1\end{pmatrix} + \frac{1}{2\theta_0-1}\begin{pmatrix} H_{(1)}^{(2)} \\ H_{(2)}^{(2)}\end{pmatrix} z + \dots\right\}, \tag{3.10b}$$

where,

$$H_{(1)}^{(1)} \doteq -y(v-\theta_0-\theta_\infty) - vt + (1 + 2\theta_0-v)(t + 2vy),$$

$$H^{(2)}_{(1)} \doteq \frac{uy}{2}\left[t + \frac{2(1-v)}{y} - \frac{y}{(2\Theta_0 - v)}\left(v - \Theta_0 - \Theta_\infty - \frac{t(1-v)}{y}\right)\right],$$

$$\sigma(t) \doteq \int^t dt' \, \frac{2v}{y} \, .$$

(Expressions for $H^{(1)}_{(2)}$, $H^{(2)}_{(2)}$ may also be given, but are not necessary for our discussion). The multiplicative constants with respect to z in (3.10) are fixed by the requirement that (3.10) also satisfy (2.7b). We note that when $\Theta_0 = n/2$ there will be, in general, a logarithmic term and the two linearly independent solutions have the form:

$$Y_0^{(1)}(z), \quad Y_0^{(2)}(z) = J(\ln z)Y_0^{(1)} + z^{-\Theta_0}\,\hat{Y}_0^{(2)}(z), \quad \Theta_0 = \frac{n}{2}\,, \qquad (3.11)$$

where $\hat{Y}_0^{(2)}$ is a polynomial in z and J is a complex constant. Equations (3.10) imply $Y_0(ze^{2i\pi}) = Y_0(z)e^{2i\pi D_0}$. Similarly, equations (3.10) and (3.11) imply (3.4).

2. Analysis near $z = \infty$.

The two linearly independent formal solutions $Y_\infty(z) = (Y_\infty^{(1)}(z), Y_\infty^{(2)}(z))$ of (2.7a) have the expansions:

$$Y_\infty^{(1)}(z) = \left\{\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-K\\ \frac{1}{u}(v-\Theta_0-\Theta_\infty)\end{pmatrix}\frac{1}{z} + \ldots\right\}\left(\frac{1}{z}\right)^{\Theta_\infty} e^{q} \qquad (3.12a)$$

$$Y_\infty^{(2)}(z) = \left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-u/2\\ K\end{pmatrix}\frac{1}{z} + \ldots\right\}\left(\frac{1}{z}\right)^{-\Theta_\infty} e^{-q}, \qquad (3.12b)$$

where

$$K \doteq \frac{v}{y}(v - 2\Theta_0) - (v - \Theta_0 - \Theta_\infty)\left(t + \frac{y}{2}\right).$$

Suppose that $Y_\infty$ is the asymptotic expansion of $Y_1$ for large z in $S_1$. According

to the Stokes phenomenon, the asymptotic expansion of $Y_1$ in sector $S_2$ is given by $Y_\infty G$ where G is a constant matrix. Alternatively, one may introduce different solutions $Y_1,\ldots,Y_4$ such that $Y_j$ is asymptotic to $Y_\infty$ in $S_j$. Then, for example, since both $Y_1$, $Y_2$ solve (2.7a) it follows that $Y_2 = Y_1G_1$, where $G_1$ is a constant (with respect to z) non-singular matrix; using the asymptotic expansions of $Y_1$, $Y_2$ it can be shown that $G_1$ has the form given in (3.6 [54]. Similarly we take $Y_5 \sim Y_\infty$ in $S_5$, where $S_5$: $2\pi - \frac{\pi}{4} \leq \arg z < 2\pi + \frac{\pi}{4}$, then $Y_5 = Y_4G_4$. Birkof [54] has related $Y_5$ and $Y_1$: $Y_1$ and $Y_5$ are defined for all z, however, they both tend to $Y_\infty$ only in $S_1$ and $S_5$ respectively:

$$Y_1(z_1) \sim \hat{Y}_\infty(z_1)e^{Q(z_1)}\left(\frac{1}{z_1}\right)^{D_\infty}, \quad z_1 \text{ in } S_1; \quad Y_5(z_5) \sim \hat{Y}_\infty(z_5)e^{Q(z_5)}\left(\frac{1}{z_5}\right)^{D_\infty}, \quad z_5 \text{ in } S_5.$$

If $z_1$ is in $S_1$ then $z_1e^{2i\pi}$ is in $S_5$. Thus

$$Y_5(z_1e^{2i\pi}) \sim \hat{Y}_\infty(z_1e^{2i\pi})e^{Q(z_1e^{2i\pi})}\left(\frac{1}{z_1e^{2i\pi}}\right)^{D_\infty} = \hat{Y}_\infty(z_1)e^{Q(z_1)}\left(\frac{1}{z_1}\right)^{D_\infty}e^{-2i\pi D_\infty},$$

where we have used that $\hat{Y}_\infty(z)$ is holomorphic at $z=\infty$ and $Q(z_1e^{2i\pi}) = Q(z_1)$. Thus

$$Y_1(z) = Y_5(ze^{2i\pi})e^{2i\pi D_\infty}. \tag{3.13}$$

Hence

$$Y_5(z) = Y_4(z)G_4, \quad \text{or} \quad Y_5(ze^{2i\pi}) = Y_4(ze^{2i\pi})G_4, \quad \text{or } Y_1(z) = Y_4(ze^{2i\pi})G_4e^{2i\pi D_\infty}.$$

3. Connecting $Y_0$ and $Y_1$.

Since both $Y_0$ and $Y_1$ satisfy (2.7a), $Y_1 = Y_0E_0$ and $\det E_0 = 1$, since $\det Y_1 = \det Y_0 = 1$. $E_0$ is called a connection matrix.

## 4. Consistency.

We first note that if the solution matrices $Y_1$, $Y_2$ are related via the constant matrix C, and if $M_1$, $M_2$ are their monodromy matrices about the same regular singular point then

$$Y_1 = Y_2 C \implies M_1 = C^{-1} M_2 C; \tag{3.14}$$

this because:

$$Y_1(ze^{2i\pi}) = Y_2(ze^{2i\pi})C = Y_2(z)M_2C = Y_1C^{-1}M_2C.$$

Equations (3.5) imply $Y_1(z) = Y_4(ze^{2i\pi})G_4M_\infty = Y_3(ze^{2i\pi})G_3G_4M_\infty \ldots =$

$= Y_1(ze^{2i\pi})(\prod_{j=1}^{4} G_j)M_\infty$. But since $M_0$ is the monodromy matrix of $Y_0$ and $Y_1 = Y_0E_0$, then $E_0^{-1}M_0E_0$ is the monodromy matrix of $Y_1$. Thus $Y_1(z) = Y_1(z)E_0^{-1}M_0E_0(\prod_{j=1}^{4} G_j)M_\infty$, which implies (3.9).

<u>Remark 3.1.</u>

(i) One has two choices: either to consider four different fundamental solutions $Y_1,\ldots,Y_4$ such that $Y_j \sim Y_\infty$ as $|z| \to \infty$, z in $S_j$, or to consider one fundamental solution Y, but then $Y \sim \hat{G}_jY_\infty$ as $|z| \to \infty$, z in $S_j$. We intend to use these $Y_j$'s to formulate a RH problem, hence it is important to have the same behavior at infinity, that is why we choose four different solutions.

(ii) The solutions $Y_1,\ldots,Y_4$ are defined in the whole complex z plane and the relationships (3.5) are valid everywhere. However, in order to formulate a RH problem we will restrict the domain of the $Y_j$'s only in the sectors that their asymptotic behavior is $Y_\infty$, thus we will use (3.5) only on the

rays $C_1,\ldots,C_4$ (see Figure 3.1).

Proposition 3.2.

(i) The monodromy data, MD, given by (3.8) and defined in Proposition 3.1 are time-invariant.

(ii) All of the MD can be expressed in terms of two of them. This follows from:

1. det $E_0 = 1$.
2. Equation (3.9).
3. If $Y$ solves (2.7) with $y$ satisfying PIV, then $\bar{Y} \doteq R^{-1}YR$, $R \doteq \mathrm{diag}(r^{1/2}, r^{-1/2})$, where $r$ is an arbitrary complex constant, also solves (2.7) with $y$ satisfying PIV. The Stokes matrices $G_j$ and the connection matrix $E_0$ are transformed to $\bar{G}_j \doteq R^{-1}G_jR$, $\bar{E}_0 \doteq R^{-1}E_0R$, i.e.

$$\bar{a} = ra,\quad \bar{b} = b/r,\quad \bar{c} = rc,\quad \bar{d} = d/r,\quad \bar{\alpha}_0 = \alpha_0,\quad \bar{\beta}_0 = \beta_0/r,$$

$$\bar{\gamma}_0 = r\gamma_0,\quad \bar{\delta}_0 = \delta_0. \tag{3.15}$$

Thus $r$ may be chosen to eliminate one parameter, e.g. $r = \beta_0$.

4. Changing the arbitrary integration constant of $\sigma(t)$ (see (3.10) ) amounts to multiplying $Y_0^{(1)}(z)$, $Y_0^{(2)}(z)$ by the arbitrary complex constants $p$ and $p^{-1}$ respectively. This maps $E_0$ to $\hat{E}_0 \doteq PE_0$, $P \doteq \mathrm{Diag}(p,p^{-1})$, i.e.

$$\hat{\alpha}_0 = p\alpha_0,\quad \hat{\beta}_0 = p\beta_0,\quad \hat{\gamma}_0 = \frac{\gamma_0}{p},\quad \hat{\delta}_0 = \frac{\delta_0}{p}. \tag{3.16}$$

Thus $p$ may be chosen to eliminate one parameter, e.g. $p = \gamma_0$.

(iii) Equation (3.9) implies

$$(1+bc)e^{2i\pi\theta_\infty} + [ad + (1+cd)(1+ab)]e^{-2i\pi\theta_\infty} = 2\cos 2\pi\theta_\infty. \tag{3.17}$$

Proof.

(i) Similar to the proof given in [14].

(ii) $\bar{Y}$ solves (2.7) iff $\bar{y} = y$, $\bar{v} = v$, $\bar{u} = ur^{-1}$ which is consistent with (2.8). Parts (ii) 3 and (iii) are straightforward. Equations (3.15), (3.16) may be chosen to fix two of the entries of the connection matrix. Then det $E_0 = 1$ and equation (3.9) imply the rest of the MD in terms of two of them.

## 3.2. The Inverse Problem.

In what follows we formulate a RH problem for the case that $0 \leq \Theta_0 < 1$, $0 \leq \Theta_\infty < 1$. This assumption leads to a regular RH problem. The general case follows by considering the results of this section and of §3.3.

In what follows we shall, for convenience of notation, consider RH problems along suitable rays. Actually, in order to satisfy convergence criteria these rays must be deformed appropriately. The deformation process is to connect at large values of z the rays to "asymptotic" curves defined by Re $q(z,t) = 0$. These asymptotic curves tend to the straight line rays (i.e. Re $q(z,t=0)$) for $|z| \to \infty$.

An alternative procedure which we anticipate to be equivalent (but one which we have not seriously considered) is to deform the rays by a sufficiently small angle $\epsilon$ into the region Re $q(z,t) < 0$. In this case the RH problem has jump matrices which rapidly tend to unity as $|z| \to \infty$. We expect the limit as $\epsilon \to 0$ of this deformed RH problem should tend to the solution of the RH problem discussed above.

Theorem 3.1.

Consider the following matrix, regular, homogeneous RH problem along the four rays $C_1,\ldots,C_4$ (Figure 3.1): Determine the sectionally holomorphic function $\Psi(z)$, $\Psi(z) = \Psi_j(z)$ if z is in $S_j$, $j = 1,\ldots,4$, from the following conditions:

1. $\Psi_j$ satisfy the jump conditions

$$\Psi_2(\zeta) = \Psi_1(\zeta)g_1(\zeta),\ \Psi_3(\zeta) = \Psi_2(\zeta)g_2(\zeta),\quad \Psi_4(\zeta) = \Psi_3(\zeta)g_3(\zeta),$$

$$\Psi_1(\zeta) = \Psi_4(\zeta e^{2i\pi})g_4(\zeta) \tag{3.18}$$

along the rays $C_2$, $C_3$, $C_4$, $C_1$ respectively, where

$$g_j \doteq e^{Q}G_j e^{-Q},\quad j = 1,2,3,\quad g_4 \doteq e^{Q}G_4 e^{-Q}M_\infty . \tag{3.19}$$

2. $$\Psi_j(z) \sim \left(\frac{1}{z}\right)^{D_\infty}\left(I + O\left(\frac{1}{z}\right)\right) \quad \text{as} \quad |z| \to \infty ,\quad \text{in } S_j. \tag{3.20}$$

3. $\Psi(z)$ has at most an integrable singularity at the origin with a monodromy matrix given by

$$\Psi_1(ze^{2i\pi}) = \Psi_1(z)E_0^{-1}M_0E_0,\qquad z \to 0. \tag{3.21}$$

In the above, $G_j$, $Q$, $M_\infty$, $D_\infty$, $M_0$ are defined in Proposition 3.1.

4. The monodromy data MD, given by (3.8), satisfy the properties given in Proposition 3.2 (ii). Then:

(i) The above RH problem is discontinuous both at the origin and at infinity. Actually

$$\prod_{j=1}^{4} g_j \sim E_0^{-1}M_0^{-1}E_0,\quad z \to 0;\qquad \prod_{j=1}^{4} g_j \sim M_\infty,\quad z \to \infty . \tag{3.22}$$

(ii) To obtain the solution of the above RH problem consider the following RH problem along the contour $C_1 + C_3$: Determine the

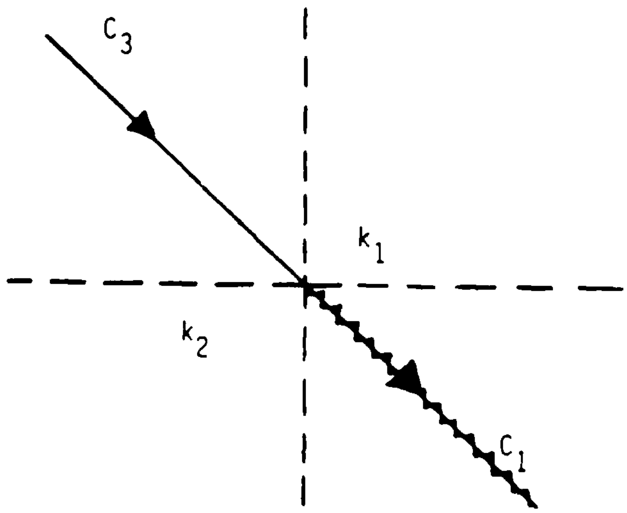


Figure 3.2

sectionally holomorphic function $k(z)$, $k(z) = k_1(z)$ if $z$ is in $S_1 + S_2$, $k(z) = k_2(z)$ if $z$ is in $S_3 + S_4$, from the following conditions:

1. $k_j$ satisfy the jump condition

$$k_1 = k_2 \begin{cases} h\begin{pmatrix} 1 & de^{2q} \\ 0 & -a/c \end{pmatrix} M_\infty h^{-1} & \text{on } C_1, \\ h\begin{pmatrix} 1 & -be^{2q} \\ 0 & -a/c \end{pmatrix} h^{-1} & \text{on } C_3 \end{cases} \qquad h(z) \doteq \begin{pmatrix} 1 & 0 \\ a\rho(z) & 1 \end{pmatrix},$$

$$\rho(z) \doteq -\frac{1}{2\pi i} \int_{C_2+C_4} \frac{d\zeta e^{-2q(\zeta)}}{\zeta - z} . \tag{3.23}$$

(If $h_1$, $h_2$ denote $h$ in $S_1 + S_4$ and $S_2 + S_3$ respectively then $h = h_1$ on $C_1$, $h = h_2$ on $C_3$).

2. $$k(z) \sim \left(\frac{1}{z}\right)^{D_\infty}\left(I + O\left(\frac{1}{z}\right)\right) \quad \text{as} \quad |z| \to \infty . \tag{3.24}$$

3. $k(z)$ has at most an integrable singularity at the origin with a monodromy matrix given by

$$k(ze^{2i\pi}) = k(z)h_1(0)E_0^{-1}M_0E_0h_1^{-1}(0), \quad z \to 0. \tag{3.25}$$

The above RH problem is discontinuous both at the origin and at infinity. Actually if $g_{k_1} g_{k_3}$ denote the jump matrices along $C_1$, $C_3$ respectively then

$$g_{k_3}^{-1}g_{k_1} \sim h_1(0)E_0^{-1}M_0^{-1}E_0h_1^{-1}(0), \quad z \to 0; \quad g_{k_3}^{-1}g_{k_1} \sim M_\infty, \quad z \to \infty. \tag{3.26}$$

However, the above RH problem can be mapped to a continuous one using the

the auxiliary functions

$$\left(\frac{z}{z\pm 1}\right)^{\pm\Theta_0}, \qquad \left(\frac{1}{z\pm 1}\right)^{\pm\Theta_\infty}, \tag{3.27}$$

to remove the above singularities (see Appendix A).

$\Psi$ is related to k via:

$$\Psi = kh \text{ if } z \text{ in } S_1+S_2;\ \Psi = khM,\ M \doteq \mathrm{Diag}(1,-a/c) \text{ if } z \text{ in } S_3+S_4 \tag{3.28}$$

(i.e. $\Psi_1 = k_1h_1$, $\Psi_2 = k_1h_2$, $\Psi_3 = k_2h_1M$, $\Psi_4 = k_2h_1M$).

Proof.

(i) We first note that the product of the jump functions $g_j$ at an intersection point, determines the nature of the singularity of the function $\Psi(z)$ at this point. For the sake of simplicity assume that $\Psi$ is scalar and that it behaves like $z^\nu$ as $z \to 0$, in $S_1$. Then $\Psi_2 \sim z^\nu g_1(0),\ldots,$ $\Psi_4 \sim z^\nu g_1(0)g_2(0)g_3(0)$, this implies $e^{-2i\pi\nu} \prod_{j=1}^{4} g(0)$. Conversely, if $\prod_{j=1}^{4} g_j(0) = e^{-2i\pi\nu}$, then $\Psi \sim z^\nu$ as $z \to 0$. This analysis is unique within the transformation $\nu \to \nu +$ integer and $z^\nu \to E^{-1}z^\nu E$, and its generalization to the case that $\Psi$ is a matrix and/or the intersection point is infinity is straightforward. In what follows we take the above integer to be zero since we are only allowing $\Psi$ to have an integrable singularity.

Equation(3.19) implies

$$\prod_{j=1}^{4} g_j \sim \left(\prod_{j=1}^{4} G_j\right)M_\infty = E_0^{-1}M_0^{-1}E_0, \quad \text{as} \quad z \to 0;$$

and hence $\Psi(z) \sim \mathrm{Diag}(z^{\Theta_0}, z^{-\Theta_0})$ $(\Theta_0 \neq n/2)$. Thus $\Psi(z)$ has a monodromy matrix $M_0$ at the origin, which is consistent with (3.21).

Similarly $\prod_{j=1}^{4} g_j \sim M_\infty$ as $|z| \to \infty$, which is consistent with the fact that $\Psi(z) \sim \mathrm{Diag}(z^{-\Theta_\infty}, z^{\Theta_\infty})$ as $|z| \to \infty$.

(ii) Consider the following transformations

$$\Psi_1 = k_1 h_1 A_1,\ \Psi_2 = k_1 h_2 A_2,\ \Psi_3 = k_2 h_2 A_3,\ \Psi_4 = k_2 h_1 A_4, \qquad (3.29)$$

where $A_j$, $j = 1,\dots,4$ are constant, non-singular matrices and the functions $k_1$, $k_2$, $h_1$, $h_2$ are defined in $S_1 + S_2$, $S_3 + S_4$, $S_4 + S_1$, $S_2 + S_3$ respectively:

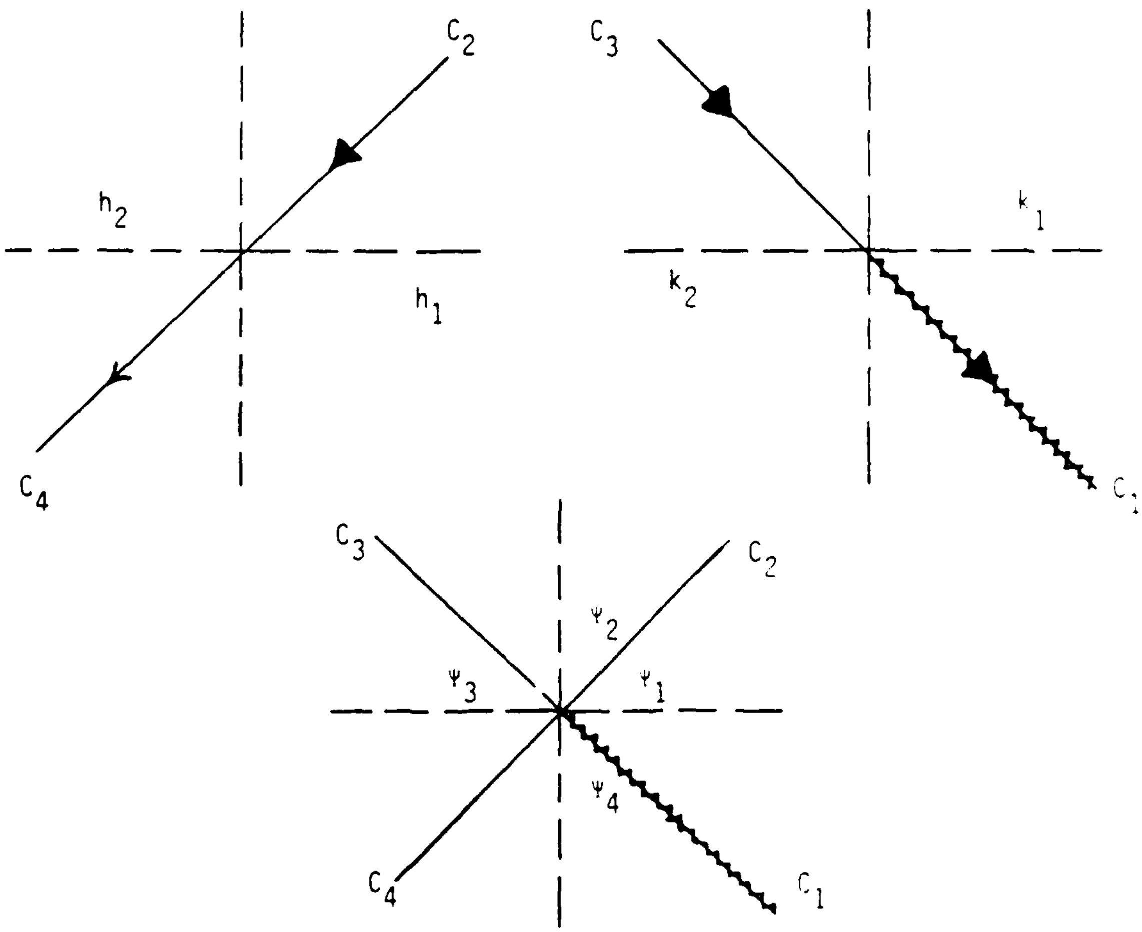


Figure 3.3

Recall that

$$C_2:\ \Psi_2 = \Psi_1 g_1, \quad C_3:\ \Psi_3 = \Psi_2 g_2, \quad C_4:\ \Psi_4 = \Psi_3 g_3, \quad C_1:\ \Psi_1 = \Psi_4 g_4. \tag{3.30}$$

Equations (3.29),(3.30) imply:

$$C_2:\ h_2 = h_1 A_1 g_1 A_2^{-1}, \qquad C_4:\ h_1 = h_2 A_3 g_3 A_4^{-1}. \tag{3.31}$$

We choose the A's in such a way that the h-RH problem is continuous both at zero and infinity.

$$\text{Continuity at zero:}\quad A_1 g_1 A_2^{-1} A_3 g_3 A_4^{-1} \sim I \quad \text{as} \quad z \to 0,$$

$$\text{Continuity at infinity:}\quad A_1 g_1 A_2^{-1} A_3 g_3 A_4^{-1} \sim I \quad \text{as} \quad z \to \infty,$$

or

$$A_1 G_1 A_2^{-1} A_3 G_3 A_4^{-1} = I, \qquad A_1 A_2^{-1} A_3 A_4^{-1} = I,$$

or

$$M \doteq A_2^{-1} A_3 = A_1^{-1} A_4, \qquad G_1 M G_3 = M. \tag{3.32}$$

Assume a, c $\neq 0$, then (3.32b) implies

$$M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & -\frac{a}{c} M_{11} \end{pmatrix}, \qquad M_{11} \neq 0. \tag{3.33}$$

Thus, (3.32) imply

$$h_1 = h_2 \begin{cases} A_2 g_1^{-1} A_1^{-1} & \text{on } C_2 \\ A_3 g_3 A_4^{-1} & \text{on } C_4 \end{cases} \quad \text{or} \quad h_1 = h_2 A_2 g_1^{-1} A_1^{-1} \quad \text{on} \quad C_2 + C_4.$$

since $A_3 g_3 A_4^{-1} = A_2(Mg_3M^{-1})A_1^{-1} = A_2 g_1^{-1} A_1^{-1}$. Hence, letting $H_1 = h_1 A_1$, $H_2 = h_2 A_2$, the above reduces to

$$C_2 + C_4 : \quad H_1 = H_2 \begin{pmatrix} 1 & 0 \\ -ae^{-2q} & 1 \end{pmatrix} . \tag{3.35}$$

Since the H-RH problem is continuous at $\infty$ we look for a solution such that $H \sim I$ as $z \to \infty$. Let $H = (H^{(1)}, H^{(2)})$, then

$$(H_1^{(1)}, H_1^{(2)}) = (H_2^{(1)}, H_2^{(2)}) - ae^{-2q}(H_2^{(2)}, 0),$$

or

$$H_1^{(2)} - H_2^{(2)} = 0, \qquad H_1^{(1)} - H_2^{(1)} = -ae^{-2q} H_2^{(2)},$$

or

$$H_1^{(2)} = H_2^{(2)} = \binom{0}{1}, \qquad H_1^{(1)} - H_2^{(1)} = -ae^{-2q}\binom{0}{1}.$$

Thus

$$H = \begin{pmatrix} 1 & 0 \\ a\rho & 1 \end{pmatrix}, \qquad \rho \text{ as in (3.23).} \tag{3.36}$$

Having obtained the solution of the H-RH problem it is straightforward to formulate the K-RH problem:

$$\begin{aligned} C_1: &\quad k_1 = k_2 h_1 A_4 g_4 A_1^{-1} h_1^{-1} \\ C_3: &\quad k_2 = k_1 h_2 A_2 g_2 A_3^{-1} h_2^{-1} \end{aligned} \qquad \text{or} \qquad k_1 = k_2 \begin{cases} h_1 A_1 M g_4 A_1^{-1} h_1^{-1} & \text{on } C_1 \\ h_2 A_2 M g_2^{-1} A_2^{-1} h_2^{-1} & \text{on } C_3 \end{cases} \tag{3.37}$$

Letting $A_1 = A_2 = I$, $M_{11} = 1$, $M_{21} = 0$, equation (3.37) reduces to (3.23a) and $H = h$. We note that the k problem inherits its singularities from the $\Psi$ problem: Consider the product of the jump matrices at infinity and

at the origin (consider (3.37) with $A_1 = A_2 = I$):

$$g_{k_3}^{-1}g_{k_1} \doteq h_2 g_2 M^{-1} h_2^{-1} h_1 M g_4 h_1^{-1}, \quad g_{k_3}^{-1}g_{k_1} \sim M_\infty, \quad |z| \to \infty,$$

$$g_{k_3}^{-1}g_{k_1} \sim h_1(0)(h_1^{-1}(0)h_2(0))G_2 M^{-1}(h_2^{-1}(0)h_1(0))MG_4 M_\infty h_1^{-1}(0), \quad z \to 0.$$

But,

$$h_1^{-1}(0)h_2(0) = \begin{pmatrix} 1 & 0 \\ -a(\rho_1(0)-\rho_2(0)) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = G_1,$$

since $\rho_1(0) - \rho_2(0) = -1$ (see (3.23c)). Also $M^{-1}G_1^{-1}M = G_3$, thus

$$g_{k_3}^{-1}g_{k_1} \sim M_\infty, \quad |z| \to \infty; \quad g_{k_3}^{-1}g_{k_1} \sim h_1(0)E_0^{-1}M_0^{-1}E_0 h_1^{-1}(0), \quad z \to 0. \tag{3.38}$$

Equation (3.38) implies that the monodromy matrix of K at the origin is $h_1(0)E_0^{-1}M_0 E_0 h_1^{-1}(0)$. This is consistent with the facts that $K_1 = \Psi_1 h_1^{-1}$ and the monodromy matrix of $\Psi_1$ is $E_0^{-1}M_0 E_0$ (see (3.14)).

Equations (3.38) imply that k has the same singular structure as $\Psi$. These singularities can be removed by using the auxiliary functions (3.27).

<u>Proposition 3.3.</u>

Let $\Psi(z)$ be the solution matrix of the inverse problem formulated in Theorem 3.1. Then y(t),

$$y(t) = -\left(\frac{1}{u}\frac{du}{dt} + 2t\right), \qquad u \doteq -2\lim_{|z|\to\infty} z\Psi_{12}(z)e^{-2q(z)}, \tag{3.39}$$

solves PIV, where $\Psi_{12}(z)$ is the upper left entry of $\Psi(z)$.

Proof.

Equation (3.12b) defines $u$ in terms of $Y$ (and hence in terms of $\tau$), and equation (2.8b) defines $y$ in terms of $u$.

### 3.3. Schlesinger Transformations.

As it was mentioned before, the case of general $\Theta_0$, $\Theta_\infty$ can be reduced to the case of $0 \le \Theta_0 < 1$, $0 \le \Theta_\infty < 1$. In this section we present the transformations which shift the values of $\Theta_0$, $\Theta_\infty$ by half-integers. Similar ideas were used by C. Cosgrove [13].

Proposition 3.4.

Let $y$ and $y'$ be solutions of PIV, equation (2.6) with $\alpha = 2\Theta_\infty - 1$, $\beta = -2\Theta_0^2$ and $\alpha' = 2\Theta'_\infty - 1$, $\beta' = -2(\Theta'_0)^2$ respectively. Let $Y$ and $Y'$ be solutions of the corresponding isomonodromic problem (2.7). Consider two sets of transformations:

$$\text{a:}\quad \begin{aligned} \Theta'_0 &= \Theta_0 + n \\ \Theta'_\infty &= \Theta_\infty + m \end{aligned} \qquad \text{b:}\quad \begin{aligned} \Theta'_0 &= \Theta_0 + \frac{2n+1}{2} \\ \Theta'_\infty &= \Theta_\infty + \frac{2m+1}{2} \end{aligned}, \qquad m,n \in \mathbb{Z}. \tag{3.40}$$

Then:

(i) The monodromy data for $Y$ and $Y'$ are the same.

(ii) The solution of the inverse problem for $Y'$ can be obtained from $Y$: $Y' = RY$. There are two cases. In particular:

$$\begin{aligned} &\text{(a)} \quad R(z) \text{ is a rational function of } z \\ &\text{(b)} \quad R(z) \text{ is } z^{1/2} \text{ times a rational function of } z. \end{aligned} \tag{3.41}$$

$$\begin{cases} \theta_0' = \theta_0 - \frac{1}{2}, \\ \theta_\infty' = \theta_\infty + \frac{1}{2}, \end{cases} \qquad R_1(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & \frac{uy}{2(v-2\theta_0)} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty)}{2(v-2\theta_0)} \end{pmatrix} z^{-1/2}, \tag{3.42a}$$

$$\begin{cases} \theta_0' = \theta_0 + \frac{1}{2}, \\ \theta_\infty' = \theta_\infty - \frac{1}{2}, \end{cases} \qquad R_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} v/y & u/2 \\ 2v/uy & 1 \end{pmatrix} z^{-1/2}, \tag{3.42b}$$

$$\begin{cases} \theta_0' = \theta_0 + \frac{1}{2}, \\ \theta_\infty' = \theta_\infty + \frac{1}{2}, \end{cases} \qquad R_3(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & uy/2v \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty)}{2v} \end{pmatrix} z^{-1/2}, \tag{3.42c}$$

$$\begin{cases} \theta_0' = \theta_0 - \frac{1}{2}, \\ \theta_\infty' = \theta_\infty - \frac{1}{2}, \end{cases} \qquad R_4(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} (v-2\theta_0)/y & u/2 \\ \frac{2}{uy}(v-2\theta_0) & 1 \end{pmatrix} z^{-1/2}. \tag{3.42d}$$

The transformations (3.42) generate all the Schlesinger transformations specified by (3.40). For example, if $R_5$, $R_6$ are defined by

$$R_5:\ \theta_0' = \theta_0,\ \theta_\infty' = \theta_\infty + 1, \qquad R_6:\ \theta_0' = \theta_0 + 1,\ \theta_\infty' = \theta_\infty .$$

Then

$$R_5 = R_1 R_3, \quad R_6 = R_2 R_4. \tag{3.43}$$

The above transformations naturally induce transformations mapping solutions of PIV to solutions of PIV with different values of the parameter. For example $R_2$ implies

$$y' = \frac{u}{u'}\left[2\frac{v}{y}\left(t - \frac{v}{y}\right) + v + \theta_0 - \theta_\infty + 1\right], \quad u' = u\left(\frac{v}{y} - \frac{y}{2} - t\right),$$

$$v' = -\frac{2v}{y}\left(t - \frac{v}{y}\right) - (v - \theta_0 - \theta_\infty), \tag{3.44}$$

where $\Theta_0'$, $\Theta_\infty'$ are defined in (3.42b) in terms of $\Theta_0$, $\Theta_\infty$.

Proof.

We first note that equation (3.17) is invariant under the transformations $\Theta_0 \to \Theta_0'$, $\Theta_\infty \to \Theta_\infty'$ iff $\Theta_0'$, $\Theta_\infty'$ are given by equations (3.40). Then Propositions 3.1, 3.2 imply that the solvability of the inverse problem for Y' is equivalent to that of Y. Under the transformations a, b, $M_\infty$ is mapped to $M_\infty'$ where $M_\infty' = M_\infty$, $M_\infty' = -M_\infty$ respectively. Thus, assuming $Y' = RY$, $R = R_j$ if $z$ in $S_j$, then the jump conditions imply a RH problem for R:

$$\begin{aligned} &\text{a:} \quad R_{j+1} = R_j \quad \text{on } C_{j+1} &&\text{b:} \quad R_{j+1} = R_j \text{ on } C_{j+1} \quad j = 1,2,3. \\ &\qquad R_1 = R_4 \quad \text{on } C_1 && \qquad R_1(z) = -R_4(ze^{2i\pi}) \quad \text{on } C_1 \end{aligned} \tag{3.45}$$

Equations (3.45) imply (3.41); to determine completely the form of R(z), i.e. to specify the rational function of z, one uses $Y' = RY$ to obtain the following boundary conditions for R:

$$\begin{aligned} \text{a:} \quad & R(z) \sim \hat{Y}_0'(z) z^{n\Pi} \hat{Y}_0^{-1}(z) \quad \text{as} \quad z \to 0, \quad \Pi \doteq \operatorname{diag}(1,-1) \\ & R(z) \sim \hat{Y}_\infty'(z) \left(\frac{1}{z}\right)^{m\Pi} \hat{Y}^{-1}(z) \quad \text{as} \quad |z| \to \infty . \end{aligned} \tag{3.46}$$

$$\text{b:} \quad \text{as above with } n \to \frac{2n+1}{2}, \quad m \to \frac{2m+1}{2}. \tag{3.47}$$

If $y'$, $u'$, $v'$, $\Theta_0' = \Theta_0 - 1/2$, $\Theta_\infty' = \Theta_\infty + 1/2$ are the transformed quantities of $y, u, v, \Theta_0, \Theta_\infty$ under the transformation given by $R_1(z)$, i.e.

$$Y'(z;y',u',v',\Theta_0',\Theta_\infty') = R_1(z;y,u,v,\Theta_0,\Theta_\infty) Y(z;y,\ldots),$$

and if $y''$, $u''$, $v''$, $\Theta_0'' = \Theta_0' + 1/2$, $\Theta_\infty'' = \Theta_\infty' - 1/2$ are the transformed

quantities of $y'$, $u'$, $v'$, $\Theta_0'$, $\Theta_\infty'$, then from the transformation given by $R_2(z)$, i.e.

$$Y''(z;y'',u'',\ldots,\Theta_\infty'') = R_2(z;y',u',\ldots,\Theta_\infty')Y'(z;y',u',\ldots,\Theta_\infty'),$$

a tedious but straightforward computation shows that,

$$R_2(z;y'(y,u,\ldots,\Theta_\infty),\ldots)R_1(z;y,u,\ldots,\Theta_\infty) = I.$$

Similarly,

$$R_3(z;y'(y,u,\ldots,\Theta_\infty),\ldots)R_4(z;y,u,\ldots,\Theta_\infty) = I.$$

Also

$$R_1(z;y'(y,u,\ldots,\Theta_\infty),\ldots)R_3(z;y,u,\ldots,\Theta_\infty) = R_5(z),$$

$$R_2(z;y'(y,u,\ldots,\Theta_\infty),\ldots)R_4(z;y,u,\ldots,\Theta_\infty) = R_6(z),$$

where

$$\begin{cases}\Theta_0' = \Theta_0 \\ \Theta_\infty' = \Theta_\infty + 1\end{cases}, \quad R_5(z) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}z + \begin{pmatrix}0 & \dfrac{u}{(v-\Theta_0-\Theta_\infty)}\\ \dfrac{-(v-\Theta_0-\Theta_\infty)}{u} & -\dfrac{v(v-2\Theta_0)}{y(v-\Theta_0-\Theta_\infty)} + t\end{pmatrix},$$

$$\begin{cases}\Theta_0' = \Theta_0 + 1 \\ \Theta_\infty' = \Theta_\infty\end{cases}, \quad R_6(z) = I + \frac{2\Theta_0+1}{N}\begin{pmatrix}-1 & -\dfrac{uy}{2z}\\ \dfrac{2v}{uy} & 1\end{pmatrix}z^{-1},$$

$$N \doteq 2\left[t - \frac{z}{y} + \frac{y}{2z}(z - \Theta_0 - \Theta_\infty)\right].$$

Hence, the successive application of the Schlesinger transformations defined by the multiplier matrices $R_k(z)$, $k = 1,2,3,4$, maps $\Theta_0$, $\Theta_\infty$ to $\Theta_0' = \Theta_0 + n/2$, $\Theta_\infty' = \Theta_\infty + m/2$, $n,m \in \mathbb{Z}$. To obtain equation (3.44), note

that the above transformations map equation (2.7a) as follows

$$Y_z = AY \implies Y'_z = (RA + R_z)R^{-1}Y'.$$

## 3.4. Special Solutions

As it was mentioned earlier, for certain choices of the parameters $\alpha,\beta$, PIV admits rational solutions or one parameter family of solutions expressible rationally in terms of the Weber-Hermite functions. Such solutions, are naturally obtained via the RH formalism presented in §3.2. For example.

Example 3.1.

Let $\Theta_\infty = -\Theta_0$, $0 < \Theta_\infty < \frac{1}{2}$, and assume that $a = c = 0$. Then $b = -d$, $E_0 = I$, and the solution of the RH defined in Theorem 3.1 is given by

$$\Psi(z) = \begin{pmatrix} (\frac{1}{z})^{\Theta_\infty} & (\frac{1}{z})^{-\Theta_\infty} \frac{d}{2\pi i} \int_C \frac{d\zeta e^{2q(\zeta)} (\frac{1}{\zeta})^{2\Theta_\infty}}{\zeta - z} \\ 0 & (\frac{1}{z})^{-\Theta_\infty} \end{pmatrix}, \qquad (3.48)$$

where the contour C is defined in Figure 3.4 (C is below the branch cut)

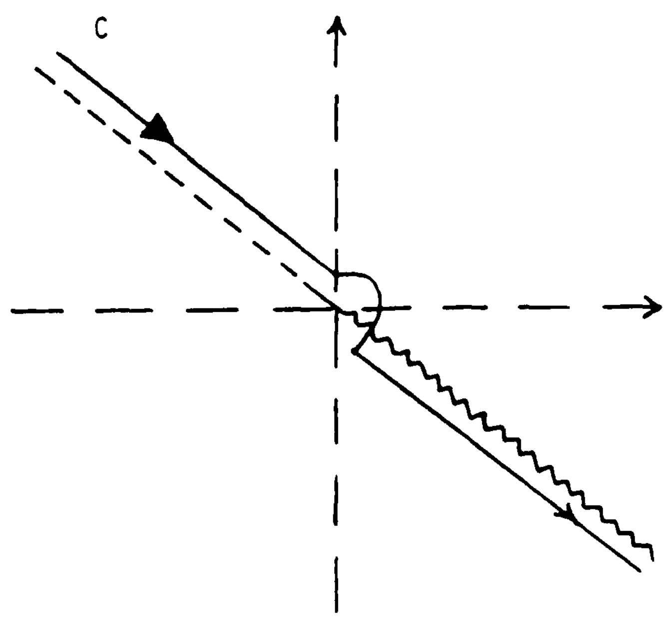


Figure 3.4.

Associated with equation (3.48), the solution $y$ of PIV is proportional to $He_{2\Theta_\infty - 1}(it)$, where $He_\nu(t)$ denotes the Weber-Hermite function [55] of parameter $\nu$.

Proof

Equation (3.17) implies that it is possible to choose $a = c = 0$ provided that $\Theta_\infty = \pm\Theta_0 + n/2$, $n \in \mathbb{Z}$. Here we will consider the case of $\Theta_\infty = -\Theta_0$ and, in view of §3.3, we assume $0 < \Theta_\infty < 1/2$. Equation (3.9) implies $b = -d$ and $E_0 = \mathrm{diag}(\alpha_0, \alpha_0^{-1})$. Using the similarity argument (equation (3.15)) we take $\alpha_0 = 1$. Thus the basic RH problem reduces to

$$\Psi^+ = \Psi^- \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}, \quad \begin{array}{llll} C_3: & \alpha = 1, & \beta = de^{2q} & \gamma = 1 \\ C_1: & \alpha = e^{2i\pi\Theta_\infty}, & \beta = de^{2q}e^{-2i\pi\Theta_\infty}, & \gamma = e^{-2i\pi\Theta_\infty}. \end{array} \tag{3.49}$$

Letting $\Psi = (\Psi_1, \Psi_2)$ the above reduces to

$$\Psi_1^+ = \alpha\Psi_1^-, \quad \Psi_2^+ = \gamma\Psi_2^- + \beta\Psi_1^-. \tag{3.50}$$

The solution of (3.50a) satisfying the boundary condition (3.20), is given by

$$\Psi_1(z) = \begin{pmatrix} (\frac{1}{z})^{\Theta_\infty} \\ 0 \end{pmatrix} . \tag{3.51}$$

Let us consider the homogeneous problem corresponding to (3.50),

$$\Phi_1^+ = \alpha\Phi_1^-, \quad \Phi_2^+ = \gamma\Phi_2^- .$$

Its solution (satisfying (3.20)) is

$$\Phi = \begin{pmatrix} (\frac{1}{z})^{\Theta_\infty} & 0 \\ 0 & (\frac{1}{z})^{-\Theta_\infty} \end{pmatrix}. \tag{3.52}$$

Thus, the solution of (3.50) is given by

$$\Psi(z) = \Phi(z)(I + \frac{1}{2\pi i} \int_{C_3+C_1} d\zeta \frac{f(\zeta)}{\zeta - z}), \tag{3.53}$$

where the matrix f given by $f = (0, \beta\Psi_1^-)(\Phi^+)^{-1}$, i.e.

$$f_{11} = f_{21} = f_{22} = 0, \quad f_{12} = d(\frac{1}{z})_-^{\Theta_\infty}(\frac{1}{z})_+^{\Theta_\infty} ,$$

where - and + denote the limits of z from the + $(S_1+S_2)$ and - $(S_3+S_4)$ regions respectively. Thus

$$f_{12} = de^{2q}(\frac{1}{re^{i\pi}})^{2\Theta_\infty} \text{ on } C_3, \quad f_{12} = de^{2q}(\frac{1}{re^{2i\pi}})^{2\Theta_\infty} \text{ on } C_1.$$

Hence substituting the above in (3.53) and using (3.52) we obtain (3.48).

## 4. PAINLEVÉ V

The fifth Painlevé equation (4.3) can be obtained [15] as the compatibility condition of the following linear systems of the equations

$$Y_z(z) = A(z)Y(z), \tag{4.1a}$$

$$Y_t(z) = B(z)Y(z) \tag{4.1b}$$

where,

$$A(z) = A_0 + A_1 \frac{1}{z} + A_2 \frac{1}{z-1}, \quad B(z) = B_0 + B_1 \frac{1}{t},$$

and

$$A_0 = \begin{pmatrix} t/2 & 0 \\ 0 & -t/2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} v + \frac{\theta_0}{2} & -u(v+\theta_0) \\ \frac{v}{u} & -(v+\frac{\theta_0}{2}) \end{pmatrix}, \quad A_2 = \begin{pmatrix} -w & uy(w-\frac{\theta_1}{2}) \\ -\frac{1}{uy}(w+\frac{\theta_1}{2}) & w \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & u[v+\theta_0-y(w-\frac{\theta_1}{2})] \\ \frac{1}{u}[v-\frac{1}{y}(w+\frac{\theta_1}{2})] & 0 \end{pmatrix},$$

$$w \doteq v + \frac{1}{2}(\theta_0 + \theta_\infty).$$

The compatibility condition of (4.1) gives:

$$t\frac{dy}{dt} = ty - 2v(y-1)^2 - \frac{1}{2}(y-1)[(\theta_0-\theta_1+\theta_\infty)y - (3\theta_0+\theta_1+\theta_\infty)], \tag{4.2a}$$

$$t\frac{dv}{dt} = yv[v + \frac{1}{2}(\theta_0-\theta_1+\theta_\infty)] - \frac{1}{y}(v+\theta_0)[v + \frac{1}{2}(\theta_0+\theta_1+\theta_\infty)], \tag{4.2b}$$

$$t\frac{du}{dt} = u\{-2t-\theta_0+y[v + \frac{1}{2}(\theta_0-\theta_1+\theta_\infty)] + \frac{1}{y}[v + \frac{1}{2}(\theta_0+\theta_1+\theta_\infty)]\}. \tag{4.2c}$$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{4.3}$$

with,

$$\alpha = \frac{1}{2}\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0-\theta_1-\theta_\infty}{2}\right)^2, \quad \gamma = 1-\theta_0-\theta_1, \quad \delta = -\frac{1}{2}. \tag{4.4}$$

The general fifth Painlevé equation with non-zero $\delta$ is reduced to the case $\delta = -1/2$ by scaling; the case $\delta = 0$ may be transformed to the third Painlevé equation which will be considered elsewhere.

## 4.1. The Direct Problem

### Proposition 4.1

Let $Y_{(0)}$, $Y_{(1)}$ be the solutions of (4.1a) analytic in the neighbor-

hood of $z = 0$ and $z = 1$ respectively and normalized by the requirements that $\det Y_{(0)} = \det Y_{(1)} = 1$, and that $Y_{(0)}$, $Y_{(1)}$ also solve (4.1b). Let $Y_1$, $Y_2$ be solutions of (4.1a) analytic in the neighborhood of infinity such that $\det Y_1 = \det Y_2 = 1$ and $Y_j \sim Y_\infty$ as $|z| \to \infty$ in $S_j$, where $Y_\infty$ is the formal solution matrix of (4.1a) in the neighborhood of infinity and

$$S_1: -\frac{\pi}{2} \leq \arg z < \frac{\pi}{2}, \qquad S_2: \frac{\pi}{2} \leq \arg z < \frac{3\pi}{2}. \tag{4.5}$$

The contours $C_1$, $C_2$, $C_3$ are defined by $\arg z = -\frac{\pi}{2}$, $\arg z = \frac{\pi}{2}$, $\arg z = 0$ and $0 \leq \mathrm{Re}\, z \leq 1$ respectively

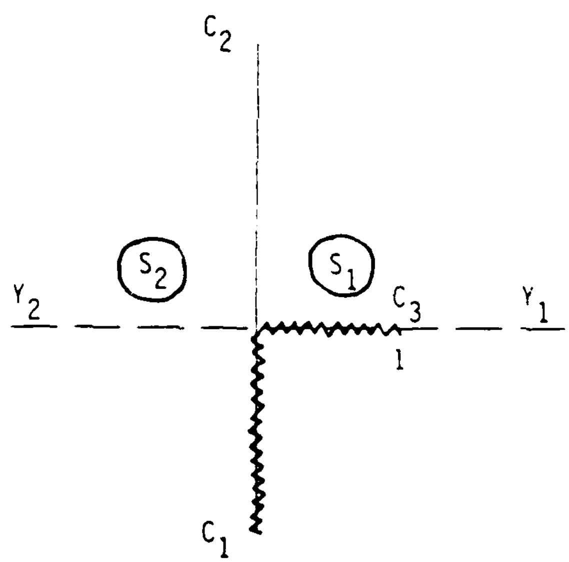


Figure 4.1

Then the analytic functions $Y_{(0)}$, $Y_{(1)}$, $Y_1$, $Y_2$ satisfy:

(i) $\quad Y_{(0)}(z) \sim \hat{Y}_{(0)}(z) z^{D_0}$ as $z \to 0$, $D_0 \doteq \mathrm{Diag}(\frac{\theta_0}{2}, -\frac{\theta_0}{2})$, $\theta_0 \neq n$, (4.6)

where $\hat{Y}_{(0)}(z)$ is holomorphic at $z = 0$ ($Y_{(0)}$ has a logarithmic singularity if $\theta_0 = n$).

(ii) $Y_{(1)}(z) \sim \hat{Y}_{(1)}(z)(z-1)^{D_1}$ as $z \to 1$, $D_1 \doteq \mathrm{Diag}(\frac{\Theta_1}{2}, -\frac{\Theta_1}{2})$, $\Theta_1 \neq n$, (4.7)

where $\hat{Y}_{(1)}(z)$ is holomorphic at $z = 1$ ($Y_{(1)}$ has a logarithmic singularity if $\Theta_1 = n$).

(iii) $Y_j(z) \sim \hat{Y}_\infty(z)e^{Q(z)}(\frac{1}{z})^{D_\infty}$ as $|z| \to \infty$ in $S_j$, $D_\infty \doteq \mathrm{Diag}(\frac{\Theta_\infty}{2}, -\frac{\Theta_\infty}{2})$, $j = 1,2$,

$$Q(z) \doteq \mathrm{Diag}(q,-q), \tag{4.8}$$

$q \doteq \frac{xt}{2}$, $\hat{Y}_\infty(z)$ is holomorphic at $z = \infty$.

(iv) $$Y_{(0)}(ze^{2i\pi}) = Y_{(0)}(z)M_0, \quad \text{as } z \to 0, \quad M_0 \doteq \begin{pmatrix} e^{i\pi\Theta_0} & 2i\pi J_0 e^{i\pi\Theta_0} \\ 0 & e^{-i\pi\Theta_0} \end{pmatrix}, \tag{4.9}$$

$J_0 = 0$ if $\Theta_0 \neq n$, $J_0 = 1$ if $\Theta_0 = n$.

(v) $$Y_{(1)}(ze^{2i\pi}) = Y_{(1)}(z)M_1, \quad \text{as } z \to 1, \quad M_1 \doteq \begin{pmatrix} e^{i\pi\Theta_1} & 2i\pi J_1 e^{i\pi\Theta_1} \\ 0 & e^{-i\pi\Theta_1} \end{pmatrix}, \tag{4.10}$$

$J_1 = 0$ if $\Theta_1 \neq n$, $J_1 = 1$ if $\Theta_1 = n$.

(vi) $$Y_2(z) = Y_1(z)G_1, \quad Y_1(z) = Y_2(ze^{2i\pi})G_2M_\infty, \tag{4.11}$$

where

$$G_1 \doteq \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 \doteq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad M_\infty \doteq \mathrm{Diag}(e^{i\pi\Theta_\infty}, e^{-i\pi\Theta_\infty}) . \tag{4.12}$$

(vii) $$Y_1 = Y_{(0)}E_0, \quad \det E_0 = 1; \quad Y_1 = Y_{(1)}E_1, \quad \det E_1 = 1. \tag{4.13}$$

(viii) Let $Y_1^+$, $Y_1^-$ dentoe the limits of $Y_1$ as it approaches $C_3$ from above and below respectively. Then

$$Y_1^+ = Y_1^- E_1^{-1} M_1 E_1 . \tag{4.14}$$

Furthermore, the parameters

$$MD \doteq \{a, b, \alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1\},$$

i.e. the entries of $G_1$, $G_2$, $E_0$, $E_1$, satisfy the following consistency conditon,

$$\text{(ix)} \qquad G_1 G_2 M_\infty = E_0^{-1} M_0^{-1} E_0 E_1^{-1} M_1^{-1} E_1. \tag{4.15}$$

Proof.

1. Analysis near $z = 0$:

Let $Y_{(0)} = (Y_{(0)}^{(1)}, Y_{(0)}^{(2)})$, then for $|z| < 1$ we find:

$$Y_{(0)}^{(1)}(z) = z^{\Theta_0/2} \frac{v}{u\Theta_0} e^{-\sigma_0} \left\{ \begin{pmatrix} \frac{u}{v}(v+\Theta_0) \\ 1 \end{pmatrix} + \begin{pmatrix} K_0^{(1)} \\ K_0^{(2)} \end{pmatrix} z + \dots \right\}, \quad \Theta_0 \neq n \tag{4.16a)}$$

$$Y_{(0)}^{(2)}(z) = z^{-\theta_0/2} e^{\sigma_0} \left\{ \begin{pmatrix} u \\ 1 \end{pmatrix} + \begin{pmatrix} L_0^{(1)} \\ L_0^{(2)} \end{pmatrix} z + \dots \right\}, \tag{4.16b}$$

where,

$$K_0^{(2)} \doteq \frac{v-1}{1+\theta_0}\left[\left(\frac{t}{2} + w\right)\left(1 + \frac{v+\theta_0}{v-1}\right) - \frac{v+\theta_0}{yv}\left(w + \frac{\theta_1}{2}\right) + \frac{vy}{1-y}\left(w - \frac{\theta_1}{2}\right)\right],$$

$$L_0^{(2)} \doteq \frac{u(1+v)}{1-\theta_0}\left[\left(\frac{t}{2} + w\right)\left(1 + \frac{v+\theta_0}{1+v}\right) - \frac{v+\theta_0}{y(1+v)}\left(w + \frac{\theta_1}{2}\right) - y\left(w - \frac{\theta_1}{2}\right)\right],$$

$$\sigma_0(t) \doteq \int^t \left\{\frac{1}{t'}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] - \frac{1}{2}\right\} dt'.$$

Expressions for $K_0^{(1)}$, $L_0^{(1)}$ may be given, but they are not needed in the discussion. The constants with respect to $z$ multiplying the $\{\ \}$ above are fixed by the requirement that equations (4.16) satisfy (4.1b). We note that when

$\Theta_0 = n$ there will be, in general, a logarithmic term and the two linearly independent solutions are

$$Y_{(0)}^{(1)}(z),\ Y_{(0)}^{(2)}(z) = J_0(\ln z)Y_{(0)}^{(1)}(z) + z^{-\Theta_0/2}\hat{Y}_{(0)}^{(2)}(z),\ \Theta_0 = n \quad (4.17)$$

where $J_0$ is a complex constant and $\hat{Y}_0^{(2)}(z)$ is a polynomial in z. Equations (4.16) imply $Y_{(0)}(ze^{2i\pi}) = Y_0(z)e^{2i\pi D_0}$. Similarly equations (4.16), (4.17) imply (4.9).

2. Analysis near $z = 1$:

Let $Y_{(1)} = (Y_{(1)}^{(1)}, Y_{(1)}^{(2)})$, then for $|z-1| < 1$ we find

$$Y_{(1)}^{(1)}(z) = (z-1)^{\theta_1/2}\,\frac{\theta_1-2w}{2\Theta_1}\,e^{-\sigma_1}\left\{\begin{pmatrix} 1 \\ \frac{1}{uy}\frac{w+\theta_1/2}{w-\theta_1/2} \end{pmatrix} + \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \end{pmatrix}(z-1) + \dots\right\},\ \Theta_1 \neq n \quad (4.18a)$$

$$Y_{(1)}^{(2)}(z) = (z-1)^{-\theta_1/2}e^{\sigma_1}\left\{\begin{pmatrix} uy \\ 1 \end{pmatrix} + \begin{pmatrix} L_1^{(1)} \\ L_1^{(2)} \end{pmatrix}(z-1) + \dots\right\}, \quad (4.18b)$$

where,

$$K_1^{(2)} \doteq \frac{1}{1+\theta_1}\left[\frac{1}{uy^2}\,\frac{w+\theta_1/2}{w-\theta_1/2}(v+\theta_0) - \frac{1}{uy}\left(\frac{t}{2}+v+\frac{\theta_0}{2}\right)\left(w+\frac{\theta_1}{2}\right)\left(\frac{1+2w}{w-\theta_1/2}\right) + \frac{v}{u}\left(1+w+\frac{\theta_1}{2}\right)\right],$$

$$L_1^{(2)} \doteq \frac{1+w-\theta_1/2}{1-\theta_1}\left[\frac{w+\theta_1/2}{y(1+w-\theta_1/2)}(v+\theta_0) + vy\left(\frac{t}{2}+v+\frac{\theta_0}{2}\right)\left(\frac{1+2w}{1+w-\theta_1/2}\right)\right].$$

$$\sigma_1(t) \doteq \int^t \left\{\frac{y}{t'}\left[v - \frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] - \frac{1}{2}\right\}dt'.$$

$$Y_{(1)}^{(1)}(z),\ Y_{(1)}^{(2)}(z) = J_1[\ln(z-1)]Y_{(1)}^{(1)}(z) + (z-1)^{-\Theta_1/2}\hat{Y}_{(1)}^{(2)}(z),\ \Theta_1 = n, \quad (4.19)$$

where $J_1$ is a complex constant and $\hat{Y}_{(1)}^{(2)}(z)$ is a polynomial of $z-1$. Expressions for $L_1^{(1)}$, $K_1^{(1)}$ may be given but they are not necessary in this discussion.

3. Analysis near $z = \infty$:

The two linearly independent formal solutins of (4.19) have the expansions

$$Y_\infty^{(1)}(z) = \left(\frac{1}{z}\right)^{\Theta_\infty/2} e^{q} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -K_\infty \\ \frac{1}{ut}[v - \frac{1}{y}(w + \frac{\theta_1}{2})] \end{pmatrix} \frac{1}{z} + \cdots \right\}, \quad (4.20a)$$

$$Y_\infty^{(2)}(z) = \left(\frac{1}{z}\right)^{-\Theta_\infty/2} e^{-q} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{u}{t}[v + \theta_0 - y(w - \frac{\theta_1}{2})] \\ K_\infty \end{pmatrix} \frac{1}{z} + \cdots \right\}, \quad (4.20b)$$

where,

$$K_\infty \doteq -\frac{1}{t}[v - \frac{1}{y}(w + \frac{\theta_1}{2})][v + \theta_0 - y(w - \frac{\theta_1}{2})] - w, \quad q \doteq \frac{zt}{2}.$$

Using similar arguments to those used in Proposition 3.1 we obtain (4.11).

Let $J_1$ denote the monodromy matrix of $Y_1$ at $z = 1$. Then $Y_1^+ = Y_1^- J_1$. However, $Y_1 = Y_{(1)}E_1$ and $M_1$ is the monodromy matrix of $Y_{(1)}$. Hence $J_1 = E_1^{-1}M_1E_1$, which implies (4.14).

(4) Consistency.

Let $J_1$ and $J_0$ denote the monodromy matrices of $Y_1$ at $z = 1$ and $z = 0$ respectively. Equations (4.11) near $z = 0$ imply:

$$Y_1^-(z) = Y_2(ze^{2i\pi})G_2M_\infty = Y_1^+(ze^{2i\pi})G_1G_2M_\infty = Y_1^+(z)J_0G_1G_2M_\infty = Y_1^-(z)J_1J_0G_1G_2M_\infty.$$

Thus $G_1G_2M_\infty = J_0^{-1}J_1^{-1}$. But $J_1 = E_1^{-1}M_1E_1$, $J_0 = E_0^{-1}M_0E_0$, thus equation (4.15) follows.

<u>Proposition 4.2.</u>

(i) The monodromy data, MD, defined in Proposition 4.1 are time invariant.

(ii) All of the MD can be expressed in terms of two of them. This follows from:

1. det $E_0$ = det $E_1$ = 1.

2. Equation (4.15).

3. If $Y$ solves (4.1) with $y$ satisfying PV, then $\bar{Y} \doteq R^{-1}YR$, $R \doteq \mathrm{Diag}(r^{1/2}, r^{-1/2})$, $r$ constant, also solves (4.1) with $y$ satisfyin PV. The Stokes matrices $G_j$ and the connection matrices $E_0$, $E_1$ are transformed to $\bar{G}_j = R^{-1}G_jR$, $\bar{E}_i = R^{-1}E_iR$, $j = 1,2$, $i = 0,1$, i.e.

$$\bar{a} = ra,\ \bar{b} = b/r,\ \bar{\alpha}_i = \alpha_i,\ \bar{\beta}_i = \beta_i/r,\ \bar{\gamma}_i = r\gamma_i,\ \bar{\delta}_i = \delta_i, \quad i = 0,1. \tag{4.21}$$

4. Changing the arbitrary integration constants $c_0$, $c_1$ amounts to multiplying $Y^{(1)}_{(i)}(z)$, $Y^{(2)}_{(i)}(z)$ by $p_i$ and $p_i^{-1}$, $i = 0,1$, respectively. This maps $E_i$ to $\hat{E}_i = P_iE_i$, $P_i \doteq \mathrm{Diag}(p_i, p_i^{-1})$, i.e.

$$\hat{\alpha}_i = p_i\alpha_i,\ \hat{\beta}_i = p_i\beta_i,\ \hat{\gamma}_i = \frac{\gamma_i}{p},\ \hat{\delta}_i = \frac{\delta_i}{p}, \quad i = 0,1. \tag{4.22}$$

<u>Proof.</u>

Similar to that of Proposition 3.2.

## <u>4.2. The Inverse Problems.</u>

In what follows we formulate a RH problem for the case that $0 \le \Theta_0 < 2$, $0 \le \Theta_1 < 2$, $0 \le \Theta_\infty < 2$. This assumption leads to a regular RH problem.

Theorem 4.1.

Consider the following matrix, regular, homogeneous RH problem along $C_1$, $C_2$, $C_3$ (Figure 4.1): Determine the sectionally holomorphic function $\Psi(z)$, $\Psi(z) = \Psi_j(z)$ if $z$ in $S_j$, $j = 1,2$ from the following conditions

1. $\Psi_j$ satisfy the jump conditions

$$\Psi_2(\zeta) = \Psi_1(\zeta)g_1(\zeta),\ \Psi_1(\zeta) = \Psi_2(\zeta)g_2(\zeta),\ \Psi_1^+(\zeta) = \Psi_1^-(\zeta)g_3, \tag{4.23}$$

along the rays $C_2$, $C_1$, $C_3$ respectively, where

$$g_1 \doteq e^Q G_1 e^{-Q},\ g_2 \doteq e^Q G_2 e^{-Q} M_\infty,\ g_3 \doteq e^Q E_1^{-1} M_1 E_1 e^{-Q}. \tag{4.24}$$

2. $$\Psi(z) \sim (\tfrac{1}{z})^{D_\infty}(I + O(\tfrac{1}{z})) \quad \text{as} \quad |z| \to \infty \quad . \tag{4.25}$$

3. $\Psi(z)$ has at most integrable singularities at $z = 0$, $z = 1$, and $z = \infty$ and the monodromy matrices of $\Psi_1$ are given by

$$E_0^{-1}M_0E_0,\quad e^{Q_1}E_1^{-1}M_1E_1e^{-Q_1},\qquad M_\infty, \tag{4.26}$$

respecitvely, where $Q_1 = Q(1)$. In the above $G_j$, $Q$, $M_\infty$, $D_\infty$, $M_0$, $M_1$ are defined in Proposition 4.1.

4. The monodromy data satisfy the properties of proposition 4.2(ii).

Then:

(i) The above RH problem is discontinuous at $z = 0, 1, \infty$. Actually

$$\prod_{j=1}^{3} g_j \sim E_0^{-1}M_0^{-1}E_0,\ z\to 0;\ g_1g_2 \sim M_\infty,\quad z \to \infty;\quad g_3^{-1} \sim e^{Q_1}E_1^{-1}M_1^{-1}E_1e^{-Q_1},\ z \to 1. \tag{4.27}$$

(ii) To obtain the solution of the above RH problem consider the following RH problem along the contour $C_0$ defined by Im $z = 0$:

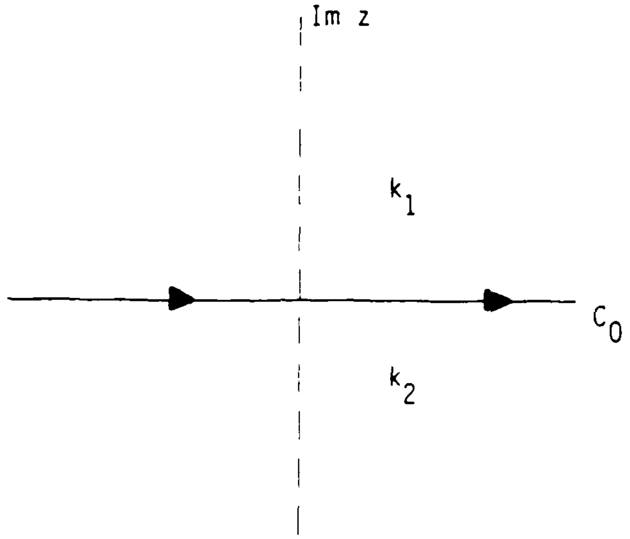


Figure 4.2

Determine the sectionally holomorphic function $k(z)$, $k(z) = k_1(z)$ if Im $z \geq 0$, $k(z) = k_2(z)$ if Im $z \leq 0$, from the following conditions:

1. $k_1$, $k_2$ satisfy the jump condition

$$C_0: \quad k_1 = k_2 \begin{cases} h_2 M h_2^{-1} & \text{Re } z < 0, \\ h_1 M M_\infty g_3 h_1^{-1} & 0 < \text{Re } z < 1 \\ h_1 M M_\infty h_1^{-1} & \text{Re } z > 1 \end{cases} \qquad M \doteq \begin{pmatrix} 0 & 1 \\ -a/b & 1 \end{pmatrix},$$

$$h(z) = \begin{pmatrix} 1 & 0 \\ a\rho(z) & 1 \end{pmatrix}, \quad \rho(z) = \frac{1}{2\pi i} \int_{C_1+C_2} \frac{d\zeta e^{\lambda\zeta t}}{\zeta - z}, \quad \lambda = 1 \text{ on } C_1,\ -1 \text{ on } C_2, \tag{4.28}$$

and $h_1$, $h_2$ denote $h$ for Re $z \geq 0$, Re $z \leq 0$ respectively.

2. $$k(z) \sim \left(\frac{1}{z}\right)^{D_\infty}\left(I + O\left(\frac{1}{z}\right)\right) \text{ as } |z| \to \infty \ . \tag{4.29}$$

3. $k(z)$ has at most integrable singularities at $z = 0$, $z = 1$ with monodromy matrices given by

$$h_1(0)E_0^{-1}M_0E_0h_1^{-1}(0), \quad h_1(1)e^{Q(1)}E_1^{-1}M_1E_1e^{-Q(1)}h_1^{-1}(1) \tag{4.30}$$

respectively.

The above RH problem is discontinuous at $z = 0, 1, \infty$. Actually if $g_{k_1}$, $g_{k_2}$, $g_{k_3}$ denote the jump matrices for Re $z < 0$, $0 <$ Re $z < 1$, Re $z > 1$ respectively then

$$g_{k_1}^{-1}g_{k_3} \sim M_\infty \text{ as } |z| \to \infty; \quad g_{k_1}^{-1}g_{k_2} \sim h_1(0)E_0^{-1}M_0^{-1}E_0h_1^{-1}(0) \text{ as } z \to 0;$$
$$(4.31)$$
$$g_{k_2}^{-1}g_{k_3} \sim h_1(1)e^{Q(1)}E_1^{-1}M_1^{-1}E_1e^{-Q(1)}h_1^{-1}(1) \text{ as } z \to 1.$$

However, the above RH problem can be mapped to a continuous one using the appropriate auxiliary functions (see Appendix B).

$\Psi$ is related to $k$ via

$$\Psi_1^+ = k_1h_1, \ \Psi_1^- = k_1h_1MM_\infty, \quad \Psi_2 = k_1h_2. \tag{4.32}$$

<u>Proof</u>

(i) The products of the jump matrices at a given point determine the nature of the singularity at this point. Equations (4.27), which follow from equations (4.24), imply: $\Psi(z) \sim \mathrm{Diag}(z^{\Theta_0/2}, z^{-\Theta_0/2})$, $z \to 0$, $\Theta_0 \neq n$ and $\Psi$ has also a log $z$ term if $\Theta_0 = n$; $\Psi(z) \sim \mathrm{Diag}((z-1)^{\Theta_1/2}, (z-1)^{-\Theta_1/2})$ $z \to 1$, and $\Psi$ has also log $(z-1)$ term in $\Theta_1 = n$; $\Psi(z) \sim \mathrm{Diag}((\frac{1}{z})^{\Theta_\infty}, (\frac{1}{z})^{-\Theta_\infty})$ as $|z| \to \infty$. Hence $\Psi(z)$ is singular at $z = 0, 1, \infty$. Assuming $0 \leq \Theta_0 < 1$, $0 \leq \Theta_1 < 1$, $0 \leq \Theta_\infty < 1$, $\Psi(z)$ is integrable at the above points.

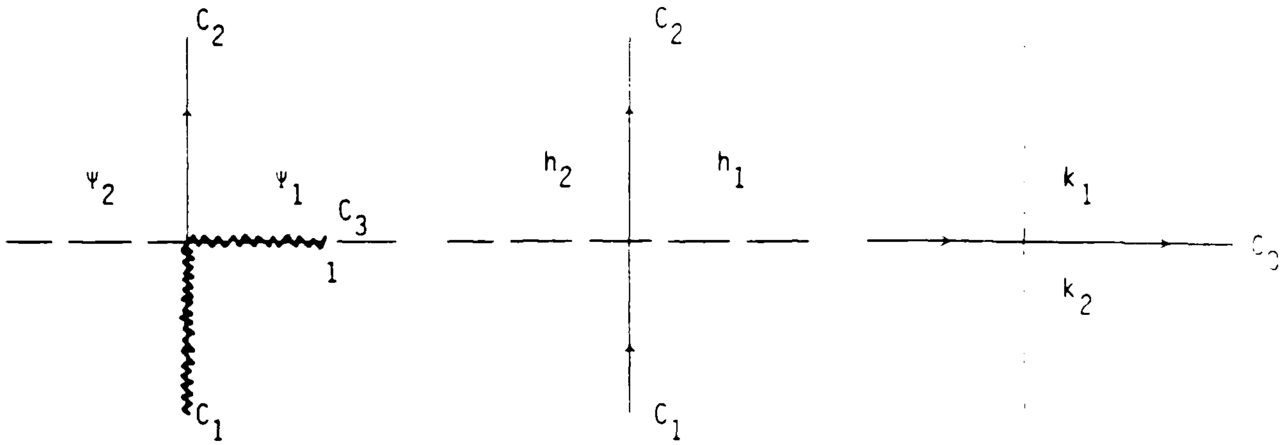


Figure 4.3

(ii) Consider the following transformations

$$\Psi_1^+ = k_1h_1A_1,\ \Psi_1^- = k_2h_1A_2,\ \Psi_2^+ = k_1h_2A_3,\ \ \Psi_2^- = k_2h_2A_4, \tag{4.33}$$

where $\Psi_2$ is $\Psi_2^+$ if Im $z > 0$ and $\Psi_2^-$ if Im $z < 0$ (clearly $\Psi_2^+ = \Psi_2^-$ in $S_2$, however we use this artificial separation in order to make the h-RH problem continuous).

$$\text{Recall}\ \ C_2\colon\ \Psi_2 = \Psi_1 g_1,\ \ C_1\colon\ \Psi_1 = \Psi_2 g_2,\ \ C_3\colon\ \Psi_1^+ = \Psi_1^- g_3 \tag{4.34}$$

Equations (4.33), (4.34) imply:

$$C_2\colon\ h_2 = h_1A_1g_1A_3^{-1},\ \ C_1\colon\ h_1 = h_2A_4g_2A_2^{-1}.$$

We choose the A's in such a way that the h-RH problem is continuous both at zero and infinity.

Continuity at zero: $A_1g_1A_3^{-1}A_4g_2A_2^{-1} \sim I \quad \text{as} \quad z \to 0,$

Continuity at infinity: $A_1g_1A_3^{-1}A_4g_2A_2^{-1} \sim I \quad \text{as} \quad z \to \infty,$

or

$$A_1G_1A_3^{-1}A_4G_2M_\infty A_2^{-1} = I, \quad A_1A_3^{-1}A_4M_\infty A_2^{-1} = I,$$

or

$$M \doteq A_3^{-1}A_4 = A_1^{-1}A_2M_\infty^{-1}, \quad G_1MG_2 = M, \quad \text{i.e. } M = \begin{pmatrix} 0 & M_{12} \\ -\frac{a}{b}M_{12} & M_{22} \end{pmatrix} \tag{4.35}$$

(assuming $a,b,M_{12} \neq 0$). Hence the h-RH problem becomes

$$h_2 = h_1\begin{cases} A_1g_1A_3^{-1} & \text{on } C_2 \\ A_1MM_\infty g_2^{-1}M^{-1}A_3^{-1} & \text{on } C_1 \end{cases} \quad \text{or} \quad h_2 = h_1A_1\begin{pmatrix} 1 & 0 \\ ae^{\lambda xt} & 1 \end{pmatrix}A_3^{-1}, \tag{4.36}$$

where $\lambda = -1$ on $C_2$, $\lambda = 1$ on $C_1$. Let $H_2 = h_2A_3$, $H_1 = h_1A_1$ and (4.36) reduces to

$$C_1 + C_2\!: \quad H_2 = H_1\begin{pmatrix} 1 & 0 \\ ae^{\lambda xt} & 1 \end{pmatrix}, \quad \lambda = 1 \text{ on } C_1, \ -1 \text{ on } C_2. \tag{4.37}$$

Since the H-RH problem is continuous at $\infty$ we look for a solution such that $H \sim I$ as $z \to \infty$.

$$(H_2^{(1)},H_2^{(2)}) = (H_1^{(1)},H_1^{(2)}) + ae^{\lambda xt}(H_1^{(2)},0),$$

or

$$H^{(2)} = \binom{0}{1}, \quad H_2^{(1)} - H_1^{(1)} = ae^{\lambda xt}\binom{0}{1}.$$

Thus

$$H = \begin{pmatrix} 1 & 0 \\ a\rho & 1 \end{pmatrix}, \quad \rho \text{ as in (4.28)}. \tag{4.38}$$

$\Psi_1^+ = \Psi_1^-$ on the Re $z$ axis, for Re $z > 1$, however equations (4.33a)

(4.33b) imply that $k_1 \neq k_2$ for Re $z > 1$ (since $A_1 \neq A_2$). Hence, although $\Psi_1$ has a discontinuity only along $C_3$ ($0 <$ Re $z < 1$), $k$ has a discontinuity along the entire positive Re $z$ axis. Similarly, equations (4.33c), (4.33d) imply that, although $\Psi_2$ is continuous in $S_2$, $k$ is discontinuous in $S_2$; we choose the negative Re $z$ axis to be the curve of the discontinuity.

Let us now formulate the K-RH problem:

$$\text{IM } z = 0:\ \Psi_2^+ = \Psi_2^-,\quad \text{Re } z < 0;\ \Psi_1^+ = \Psi_1^- g_3,\ 0 < \text{Re } z < 1;\ \text{[illegible]}$$

Using (4.32) we find

$$\text{Im } z = 0: \quad k_1 = k_2 \begin{cases} h_2 A_4 A_3^{-1} h_2^{-1} & \text{Re } z < 0 \\ h_1 A_2 g_3 A_1^{-1} h_1^{-1} & 0 < \text{Re } z < 1 \\ h_1 A_2 A_1^{-1} h_1^{-1} & \text{Re } z > 1 \end{cases} \tag{4.39}$$

Using $A_4 = A_3 M$, $A_2 = A_1 M M_\infty$ we find

$$k_1 = k_2 \begin{cases} h_2 A_3 M A_3^{-1} h_2^{-1} & \text{Re } z < 0 \\ h_1 A_1 M M_\infty g_3 A_1^{-1} h_1^{-1} & 0 < \text{Re } z < 1 \\ h_1 A_1 M M_\infty A_1^{-1} h_1^{-1}\ , & \text{Re } z > 1 \end{cases}$$

which, with the choice of $A_3 = A_1 = I$, $M_{12} = 1$, $M_{22} = 0$, reduces to (4.28). The $k$ problem inherits its singularities from the $\Psi$ problem. Consider the product of the jump matrices at the singular points:

$$g_{k_1} \doteq h_2 M h_2^{-1},\quad g_{k_2} \doteq h_1 M M_\infty g_3 h_1^{-1},\quad g_{k_3} \doteq h_1 M M_\infty h_1^{-1}.$$

Then

$$g_{k_1}^{-1} g_{k_3} \sim M_\infty \text{ as } |z| \to \infty,\quad g_{k_1}^{-1} g_{k_2} \sim h_1(0) E_0^{-1} M_0^{-1} E_0 h_1^{-1}(0) \text{ as } z \to 0,$$

$$g_{k_2}^{-1} g_{k_3} \sim h_1(1) e^{Q(1)} E_1^{-1} M_1^{-1} E_1 e^{-Q(1)} h_1^{-1}(1) \quad \text{as} \quad z \to 1. \qquad (4.40)$$

Equations (4.40a), (4.40c) are obvious. To obtain equation (4.40b) note:

$$h_1^{-1}(0) h_2(0) = \begin{pmatrix} 1 & 0 \\ a(\rho_2(0)-\rho_1(0)) & 1 \end{pmatrix} = G_1, \quad \text{since } \rho_2(0)-\rho_1(0) = 1$$

(see (4.38)).

Hence,

$$g_{k_1}^{-1} g_{k_2} = h_2(0) M^{-1} h_2^{-1}(0) h_1(0) M M_\infty g_3(0) h_1^{-1}(0) = h_1(0) G_1 M^{-1} G_1^{-1} M M_\infty g_3(0) h_1^{-1}(0) =$$

$$= h_1(0) G_1 G_2 M_\infty E_1^{-1} M_1 E_1 h_1^{-1}(0) = h_1(0) E_0^{-1} M_0^{-1} E_0 h_1^{-1}(0), \qquad (4.41)$$

where we have used $M^{-1} G_1^{-1} M = G_2$ (see (4.35)), and (4.15).

Equation (4.40b) implies that the monodromy matrix of k at the origin is $h_1(0) E_0^{-1} M_0 E_0 h_1^{-1}(0)$. This is consistent with the facts that $k_1 = \Psi_1^+ h_1$ and the monodromy matrix of $\Psi_1$ is $E_0^{-1} M_0 E_0$. Similarly for the monodromy matrix of k at z = 1. Thus k has the same singularities as $\Psi$. These singularities can be removed by using appropriate auxiliary functions.

<u>Proposition 4.3.</u>

Let $\Psi(z)$ be the solution matrix of the inverse problem formulated in Theorem 4.1. Then y(t), which may be obtained from (4.1a), (4.2a), solves PV.

## 4.3. Schlessinger Transformations.

Proposition 4.4.

Let $y$ and $y'$ be solutions of PV equation (4.4) with $\alpha$, $\beta$, $\gamma$ and $\alpha'$, $\beta'$, $\gamma'$ respectively, where $\alpha$, $\beta$, $\gamma$ are related to $\Theta_0$, $\Theta_1$, $\Theta_\infty$ via (4.4). Let $Y$, $Y'$ be solutions of the corresponding isomonodromic problem (4.1). Consider the sets of transformations:

$$\text{a:}\ \begin{cases}\Theta_0' = \Theta_0 \pm n \\ \Theta_1' = \Theta_1 \\ \Theta_\infty' = \Theta_\infty \pm m\end{cases} \qquad \text{b:}\ \begin{cases}\Theta_0' = \Theta_0 \\ \Theta_1' = \Theta_1 \pm n \\ \Theta_\infty' = \Theta_\infty \pm m\end{cases} \qquad \text{c:}\ \begin{cases}\Theta_0' = \Theta_0 \pm n \\ \Theta_1' = \Theta_1 \pm m, \\ \Theta_\infty' = \Theta_\infty\end{cases}$$

where, $n,m$ are either even or odd integers. (4.42)

Then:

(i) The monodromy data for $Y$ and $Y'$ are the same.

(ii) The solution of the inverse problem for $Y'$ can be obtained from $Y$:

$$Y' = RY;\quad \begin{aligned} &\text{a: } R(z) \text{ is } z^{1/2} \text{ times a rational function of } z, \\ &\text{b: } R(z) \text{ is } (z-1)^{1/2} \text{ times a rational function of } z, \\ &\text{c: } R(z) \text{ is } z^{-1/2}(z-1)^{1/2} \text{ times a rational function of } z. \end{aligned} \tag{4.43}$$

In partiuclar:

$$\begin{cases}\Theta_0' = \Theta_0 + 1 \\ \Theta_1' = \Theta_1 \\ \Theta_\infty' = \Theta_\infty + 1\end{cases}, \qquad R_1(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} +$$

$$\begin{pmatrix} 1 & -\frac{u}{z}(z + \Theta_0) \\ -\frac{1}{tu}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] & [\frac{1}{v}(v + \Theta_0)][\frac{1}{t}(v - \frac{1}{y}(w + \frac{\Theta_1}{v}))] \end{pmatrix} z^{-1/2}, \tag{4.44a}$$

$$\begin{cases} \Theta_0' = \Theta_0 - 1 \\ \Theta_1' = \Theta_1 \\ \Theta_\infty' = \Theta_\infty - 1 \end{cases}, \qquad R_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} +$$

$$\begin{pmatrix} \frac{1}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] & -\frac{u}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] \\ -\frac{1}{u} & 1 \end{pmatrix} z^{-1/2}, \quad (4.44b)$$

$$\begin{cases} \Theta_0' = \Theta_0 + 1 \\ \Theta_1' = \Theta_1 \\ \Theta_\infty' = \Theta_\infty - 1 \end{cases}, \qquad R_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} +$$

$$\begin{pmatrix} \frac{1}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})]\frac{v}{v+\Theta_0} & -\frac{u}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] \\ -\frac{v}{u(v+\Theta_0)} & 1 \end{pmatrix} z^{-1/2}, \quad (4.44c)$$

$$\begin{cases} \Theta_0' = \Theta_0 - 1 \\ \Theta_1' = \Theta_1 \\ \Theta_\infty' = \Theta_\infty + 1 \end{cases}, \qquad R_4(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} +$$

$$\begin{pmatrix} 1 & -u \\ -\frac{1}{ut}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] & \frac{1}{t}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] \end{pmatrix} z^{-1/2}, \quad (4.44d)$$

$$\begin{cases} \Theta_0' = \Theta_0 \\ \Theta_1' = \Theta_1 + 1 \\ \Theta_\infty' = \Theta_\infty + 1 \end{cases},$$

$$R_5(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(z-1)^{1/2} + \begin{pmatrix} 1 & w_1 \\ \frac{1}{ut}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] & \frac{y}{t}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})]w_1 \end{pmatrix}(z-1)^{-1/2}$$

where $w_1 \doteq (w - \Theta_1/2)/(w + \Theta_1/2)$.

$$\begin{cases} \Theta_0' = \Theta_0 \\ \Theta_1' = \Theta_1 - 1 \\ \Theta_\infty' = \Theta_\infty - 1 \end{cases}, \qquad R_6(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(z-1)^{1/2} +$$

$$\begin{pmatrix} \frac{1}{ty}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] & -\frac{u}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] \\ -\frac{1}{uy} & 1 \end{pmatrix}(z-1)^{-1/2}, \tag{4.44f}$$

$$\begin{cases} \Theta_0' = \Theta_0 \\ \Theta_1' = \Theta_1 + 1, \\ \Theta_\infty' = \Theta_\infty - 1 \end{cases} \qquad R_7(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(z-1)^{1/2} +$$

$$\begin{pmatrix} \frac{1}{ty}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})]w_1^{-1} & -\frac{u}{t}[v + \Theta_0 - y(w - \frac{\Theta_1}{2})] \\ -\frac{1}{uy}w_1^{-1} & 1 \end{pmatrix}(z-1)^{-1/2},$$

$$\begin{cases} \Theta_0' = \Theta_0 \\ \Theta_1' = \Theta_1 - 1 \\ \Theta_\infty' = \Theta_\infty + 1 \end{cases}, \qquad R_8(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(z-1)^{1/2} +$$

$$\begin{pmatrix} 1 & -uy \\ -\frac{1}{ut}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] & \frac{y}{t}[v - \frac{1}{y}(w + \frac{\Theta_1}{2})] \end{pmatrix}(z-1)^{-1/2}. \tag{4.44h}$$

The transformations (4.44) generate all the Schlesinger transformations specified by (4.42). For example,

$$R_{j+1}R_j = I, \quad j = 1,3,5,7; \quad R_1R_7 = R_9; \quad R_2R_8 = R_{10}, \quad \text{etc.}$$

where

$$R_9:\ \Theta_0' = \Theta_0 + 1,\ \Theta_1' = \Theta_1 + 1,\ \Theta_\infty' = \Theta_\infty;$$

$$R_{10}:\ \Theta'_0 = \Theta_0 - 1,\ \Theta_1' = \Theta_1 - 1,\ \ \Theta_\infty' = \Theta_\infty. \tag{4.45}$$

The above transformations induce transformations mapping solutions of PV to solutions of PV with parameters related as in equations (4.42).

<u>Proof</u>

Equation (4.15) is invariant under the transformations $\Theta_0 \to \Theta_0'$, $\Theta_1 \to \Theta_1'$, $\Theta_\infty \to \Theta_\infty'$ iff the $\Theta$'s transform as in (4.42). It will turn out that it is sufficient to consider m = n = 1. Since $M_1, M_\infty \to \pm M_1, \pm M_\infty$, assuming $Y' = RY$, equations (4.23) imply

$$\begin{array}{llllll}
\text{a:} & R_2 = R_1^+, & \text{B:} & R_2 = R_1^+, & \text{c:} & R_2 = R_1^+ \ \text{on}\ C_2 \\
 & R_1^+ = R_1^- & & R_1^+ = -R_1^- & & R_1^+ = -R_1^- \ \text{on}\ C_3 \\
 & R_1^- = -R_2 & & R_1^- = -R_2 & & R_1^- = R_2 \ \text{on}\ C_1 .
\end{array} \tag{4.46}$$

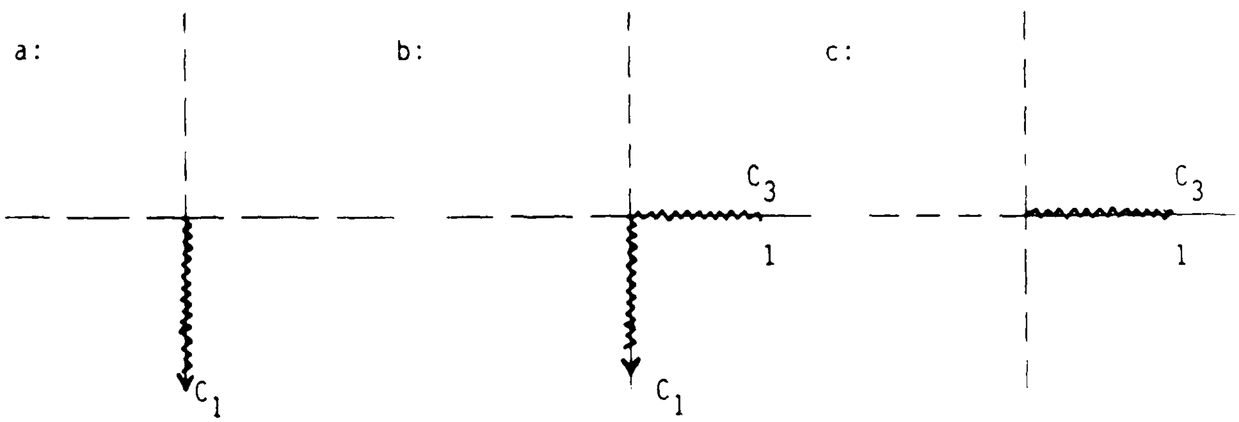


**Figure 4.4**

Equations (4.46) imply (4.43); to determine completely the form of $R(z)$, use $Y' = RY$ and the boundary conditions for $Y$ to obtain appropriate boundary conditions for $R$ (the details are given in [45]).

## 3.4. Special Solutions

**Example 4.1.**

Let $\Theta_\infty = -(\Theta_0 + \Theta_1)$, $0 < \Theta_0 < 1$, $0 < \Theta_1 < 1$ and assume that $a = 0$. Then $\beta_1 = \gamma_1 = \gamma_0 = 0$, and the solution of the RH defined in Theorem 4.1 is given by

$$\Psi(z) = \begin{pmatrix} z^{\Theta_0/2}(z-1)^{\Theta_1/2} & bz^{-\Theta_0/2}(z-1)^{-\Theta_1/2}\dfrac{1}{2\pi i}\displaystyle\int_C \frac{d\zeta e^{\zeta t}\zeta^{\Theta_0}(\zeta-1)^{\Theta_1}}{\zeta - z} \\ 0 & z^{-\Theta_0/2}(z-1)^{-\Theta_1/2} \end{pmatrix}, \tag{4.47}$$

where the contour $C$ is along $C_1$ in the $S_2$ region. Hence associated with equation (4.47), is a solution $y$ of PV which is proportional to $W_{\Theta_0,\Theta_1}(t)$, where $W$ denotes the Whittaker function [55].

To derive the above, note that in this case the basic RH problem reduces to

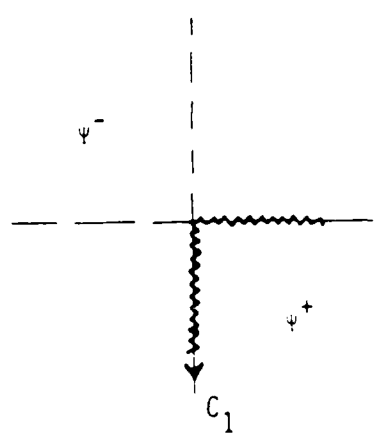


Figure 4.5

$$\Psi^+ = \Psi^- \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}; \quad \begin{array}{ll} \mathrm{Im}\, z = 0, \ \mathrm{Re}\, z > 1 & : \alpha = 1, \ \gamma = 1, \ \beta = 0 \\ \mathrm{Im}\, z = 0, \ 0 < \mathrm{Re}\, z < 1 & : \alpha = e^{-i\pi\Theta_1}, \ \gamma = e^{i\pi\Theta_1}, \ \beta = 0 \\ C_1 & : \alpha = e^{i\pi\Theta_\infty}, \ \gamma = e^{-i\pi\Theta_\infty}, \ \beta = be^{-i\pi\Theta_\infty + \zeta t} \end{array} \tag{4.48}$$

Letting $\Psi = (\Psi_1, \Psi_2)$ the above reduces to

$$\Psi_1^+ = \alpha\Psi_1^-, \quad \Psi_2^+ = \gamma\Psi_2^- + \beta\Psi_1^- \tag{4.49}$$

Equaiton (4.49a) implies

$$\Psi_1 = z^{\Theta_0/2}(z-1)^{\Theta_1/2}\binom{1}{0}. \tag{4.50}$$

The "homogeneous" version of (4.49) yields

$$\Phi = \begin{pmatrix} z^{\Theta_0/2}(z-1)^{\Theta_1/2} & 0 \\ 0 & z^{-\Theta_0/2}(z-1)^{-\Theta_1/2} \end{pmatrix}. \tag{4.51}$$

Thus the solution of (4.49) is given by

$$\Psi(z) = \Phi(z)\left(I + \frac{1}{2\pi i}\int_{C_1} \frac{d\zeta f(\zeta)}{\zeta - z}\right), \tag{4.52}$$

where $f = (0, \beta\Psi_1^-)(\Phi^+)^{-1}$, i.e.

$$f_{11} = f_{12} = f_{22} = 0, \quad f_{21} = be^{-i\pi\Theta_\infty + \zeta t}[\zeta^{\Theta_0/2}(\zeta-1)^{\Theta_1/2}]_-[\zeta^{\Theta_0/2}(\zeta-1)^{\Theta_1/2}]_+.$$

where - and + denote the limits of z in $S_2$ and $S_1$ respectively. Since $f_{21} = be^{\zeta t}(\zeta^{\Theta_0/2}(\zeta-1)^{\Theta_1/2})_-^2$ we obtain (4.47).

## APPENDIX A

In Theorem 3.1 we mapped the basic RH problem which underlies the initial value problem of PIV to a simpler RH problem, i.e. equations (3.23)-(3.26). This RH problem is discontinuous both at zero and infinity. Actually the product of the jump matrices at $z = 0$ and $z = \infty$ is given by

$$h_1(0)E_0^{-1}M_0^{-1}E_0h_1^{-1}(0), \qquad M_\infty , \tag{A.1}$$

respectively. We now map this discontinuous problem to a continuous one. The basic idea is to use appropriate discontinuous auxiliary functions such that the product of the jump matrices of the transformed problem is I. This procedure is the well known [46] so the derivation is omitted (details are given in [45]).

### Proposition A.1.

Consider the k-RH problem formulated in Theorem 3.1 and defined by equations (3.23)-(3.26). Assume $0 < \theta_0 < 1$, $0 \leq \theta_\infty < 1$, $\theta_0 \neq 1/2$. Define the sectionally holomorphic function $\Phi(z)$, $\Phi(z) = \Phi^-(z)$ if $z$ is in $S_1 + S_2$, $\Phi(z) = \Phi^+(z)$ if $z$ is in $S_3 + S_4$ as follows:

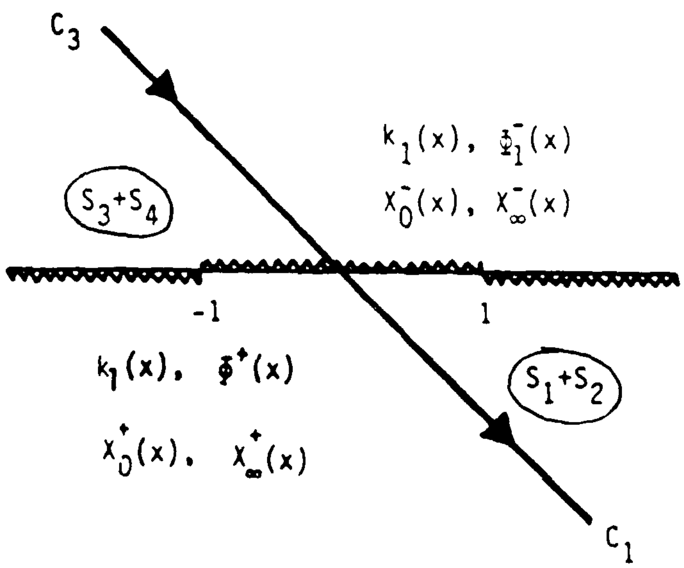


Figure A.1

$$k_1(z) = \Phi^-(z)X_\infty^-(z)R_\infty^-X_0^-(z)R_0^-, \quad z \text{ in } S_1 + S_2,$$
$$k_2(z) = \Phi^+(z)X_\infty^+(z)R_\infty^+X_0^+(z)R_0^+, \quad z \text{ in } S_3 + S_4, \tag{A.2}$$

where,

$$X_0^-(z) = \mathrm{Diag}[(\frac{z}{z+1})^{\Theta_0}, (\frac{z}{z+1})^{-\Theta_0}], \; X_0^+(z) = \mathrm{Diag}[(\frac{z}{z-1})^{\Theta_0}, (\frac{z}{z-1})^{-\Theta_0}], \tag{A.3}$$

$$X_\infty^-(z) = \mathrm{Diag}[(\frac{1}{z+1})^{\Theta_\infty}, (\frac{1}{z+1})^{-\Theta_\infty}], \; X_\infty^+(z) = \mathrm{Diag}[(\frac{1}{z-1})^{\Theta_\infty}, (\frac{1}{z-1})^{-\Theta_\infty}]. \tag{A.4}$$

The auxiliary functions $X_0$, $X_\infty$ are defined with respect to a finite branch cut from $z = -1$ to $z = 1$, and an infinite branch cut from $z = 1$ to $z = -1$ respectively; these branches are specified by

$$\lim_{|z|\to\infty} X_0^-(z) = I, \qquad \lim_{z\to 0} X_\infty^-(z) = I. \tag{A.5}$$

The constant matrices $R_0$, $R_\infty$ are defined by:

$$R_0^- \doteq E_0h_1^{-1}(0), \; R_0^+ \doteq E_0G_1G_2G_3M^{-1}h_1^{-1}(0), \; R_\infty^- \doteq (R_0^-)^{-1}, \; R_\infty^+ \doteq (R_0^+M)^{-1}, \; M = \mathrm{Diag}(1,-\frac{3}{c} \tag{A.6}$$

Then the $\Phi$-RH problem,

$$C_1+C_3{:}\; \Phi^+(\zeta) = \Phi^-(\zeta)X_\infty^{-1}(\zeta)R_\infty^-X_0^-(\zeta)R_0^-g_k(\zeta)[X_\infty^+(\zeta)R_\infty^+X_0^+(\zeta)R_0^+]^{-1}, \tag{A.7}$$

$$\Phi^-(z) \to I \quad \text{as} \quad |z| \to \infty \text{ in } S_1 + S_2,$$

where $g_k$ is the jump matrix of the k-RH problem given by (3.23), is continuous.

Remark A.1.

The case $\Theta_0 = 0$ can be handled in a similar way: In this case k

$X_0$, $X_1$ are defined with respect to finite branch cuts between $z = z_0 \doteq 1 + i$ and $z = \bar{z}_0 = 1 - i$ passing through $z = 0$ and $z = 1$ respectively; $X_\infty$ is defined with respect to an infinite branch cut between the points $z = z_0$ and $z = \bar{z}_0$. These branches are normalized by

$$\lim_{|z|\to\infty} X_0(z) = I, \quad \lim_{|z|\to\infty} X_1(z) = I, \quad \lim_{z\to 0} X_\infty(z) = X_{\infty,0}, \quad \lim_{z\to 1} X_\infty(z) = X_{\infty,1}, \tag{B.5}$$

with $[X_\infty(\infty+)]^{-1}X_\infty(\infty-) = M_\infty$. If the constant matrices are appropriately chosen then $\Phi(z)$ satisfies a continuous RH problem (the details can be found in [45]).

Remark B.1

The cases $\Theta_0 = 0$, $\Theta_1 = 0$ can be handled in a similar way. The logarithmic singularities can be removed by using

$$X(z) = \begin{pmatrix} 1 & \ln F \\ 0 & 1 \end{pmatrix},$$

where $z_0^+$, $z_0^-$, $z_1^+$, $z_1^-$ corresponds to F

$$\left(\frac{z}{z-\bar{z}_0}\right), \quad \left(\frac{z}{z-z_0}\right), \quad \left(\frac{z-1}{z-\bar{z}_0}\right), \quad \left(\frac{z-1}{z-z_0}\right) \text{ respectively.}$$

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[1] Painleve, P.: Bull. Soc. Math. Fr. 28, 214 (1900); Painleve, P.: Acta Math. 25, 1 (1902).

[2] Gambier, B.: Acta Math. 33, 1 (1909).

[3] Ince, E.L.: Ordinary differential equations (1927). New York: Dover (1956).

[4] Fuchs, R.: Math. Ann., 63, 301-321, (1907).

[5] Garnier, R.: Ann. Sci. Ec. Norm. Super., 29, 1-126 (1912).

[6] Lax, P.D.: Comm. Pure Appl. Math., 21, 467 (1968).

[7] Ablowitz, M.J., Ramani, A., Segur, H.: Lett. Nuovo Cimento 23, 333 (1978); J. Math. Phys. 21, 715 (1980); J. Math. Phys. 21, 1006 (1980).

[8] Olver, P. and McLeod, SIAM J. Math. Anal. 14, 488 (1983).

[9] Fokas, A.S., Ablowitz, M.J.: J. Math. Phys. 23, 2033 (1982).

[10] Ablowitz, M.J., Segur, H.: Phys. Rev. Lett. 38, 1103 (1977).

[11] Fokas, A.S., Leo, R.A., Martina, L., Soliani, G.: PHys. Lett. A, 115, 7 (1986).

[12] Ernst, F.: Phys. Rev. 167, 1175 (1968); Phys. Rev. 168, 1415 (1968).

[13] Cosgrove, C.: (private communication).

[14] Flaschka,H., Newell, A.C.: Commun. Math. Phys. 76, 67 (1980).

[15] Ueno, K.: Proc. Jpn. Acad. 56A, 97 (1980); 56A, 103 (1980); and Jimbo, M., Miwa, T., Ueno, K.: Physica 2D, 306 (1981); Jimbo, M., Miwa, T.: Physica 2D, 407 (1981); 4D, 47 (1981); M. JImbo, Prog. Theoret. Phys. 61, 359 (1979).

[16] Its, A.R., Novokshenov, V.Y.: The Isomonodromic Deformation Method *in the* Theory of Painleve Equations, Lect. Notes Math., ed. A. Dold and B. Eckmann, Springer-Verlag, 1191.

[17] Garnier, R.: Rend. CIrc. Math. Palenmo, 43, 155-191 (1919).

[18] Barouch, E., McCoy, B.M., Wu, T.T.: Phys. Rev. Lett. 31, 1409 (1973); Wu, T.T., McCoy,B.M., Tracy, C.A., Barouch, E.: Phys. Rev. B 13, 316 (1976).

[19] Jimbo, M., Miwa, T., Mori, Y., Sato, M.: Physica 1D, 80 (1980); Jimbo, M., Miwa, T.: Proc. Jpn. Acad., 56 A, 405 (1980).

[20] Miles, J.W.: Proc. R. Soc. London A 361, 277 (1978).

[21] McCoy, B.M., Perk, J.H.H., Schrock, R.: Nucl. Phys. B (1983) and Correlation functions of the transverse Ising chain at the critical field for large temporal and spatial separations.

[22] McCoy, B.M., Tracy, C.A., Wu, T.T.: J. Math. Phys., 18, 5, 1058, (1977).

[23] McCoy, B.M., Wu, T.T.: Nucl. Phys., B180 (FS2), 89, (1981).

[24] Creamer, D.B., Thacker, H.B., Wilkinson, D., Phys. Rev. D., 23, no. 12, 3081 (1981).

[25] Perk, J.H.H., Capel, H.W., Quispel, G.R.W., Nijhoff, F.W.: Finite-temperature correlations for the Ising chain in a transverse field. - Preprint ITP-SB-82-72 (Univ. NY, Stony Brook).

[26] Gromak, V.I., Tsegelnik, V.V.: Teor. Mat. Fiz., 55, no. 2, 189-196 (1983).

[27] Zakharov, V.E., Kuznetsov, E.A., Musher, S.L., IETP Lett., 41, no. 3, 125-127, (1985).

[28] Tajiri, M.: J. Phys. Soc. Japan, 53, no. 1, 1-4 (1984).

[29] Vaidya, H.G., Tracy, C.A.: J. Math. Phys., 20, no. 11, 2291-2312 (1979).

[30] Kevorkian, J., Cole, J.: Perturbation Methods in Applied Mathematics, Springer-Verlag, NY, 1981.

[31] Lukashevich, N.A.: Diff. Urav. 7, 1124 (1971).

[32] Gromak, V.I.: Diff. Urav. 11, 373 (1975).

[33] Lukashevich, N.A.: Author's abstract of doctoral dissertation (in Russian) Kiev, 1971.

[34] Gromak, V.I.: Diff. Urav. 12, 740 (1976).

[35] Fokas, A.S., Yortsos, Y.C.: Lett. Nuovo Cimento 30, 539-540 (1980).

[36] Erugin, N.P.: Dokl. Akad. Nark BSSR, 2 (1958).

[37] Lukashevich, N.A.: Diff. Urav. 3, 994 (1967).

[38] Lukashevich, N.A.: Diff. Urav. 3, 395 (1967).

[39] Lukashevich, N.A.: Diff. Urav. 4, 737 (1968).

[40] Lukashevich, N.A., Yablonski, A.I.: Diff. Urav. 3, 264 (1967).

[41] Fokas, A.S., Ablowitz, M.J.: Phys. Rev. Lett. 47, 1096 (1981).

[42] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Phys. Rev. Lett. 19, 1095 (1967); Comm. Pure and Appl. Math. 27, 97 (1974).

[43] Fokas, A.S., Ablowitz, M.J.: Comm. Math. Phys. 91, 331 (1983).

[44] Krein, M.G.: Usp. Mat. Nauk. 13, 3 (1958); Gohberg, I., Krein, M.G.: USP. Mat. Nauk. 13, 2 (1958);

[45] Mugan, U.: Ph.D. thesis, "On the inital value problem of some Painleve equations", Clarkson University, Dec. 1986.

[46] Gakhov, F.D.: Boundary value problems, New York: Pergamon Press 1966; Muskhelishvili, N.I.: Singular integral equations, Groningen: Noordhoff, 1953; Vekua, N.P., Systems of singular integral equations, New York: Gordon and Breach, 1967.

[47] Prossdorf, S.: Some classes of singular equations. Amsterdam: North-Holland 1978.

[48] Fokas, A.S. and Ablowitz, M.J.: IST for Multidimensional (2+1) Problems, ed. by K.B. Wolf, Springer-Verlag, Berlin, 1983.

[49] Fokas, A.S. and Papageorgiou, V.: Inverse Problems and a Unified Approach to Integrability in 1, 1+1, and 2+1 Dimensions, INS #74.

[50] Hormander, L.: Complex analysis in several variables, Van Nostrand, 1966.

[51] Schlesinger, L., J. Reine Angewandte Math., 141, 96 (1912).

[52] Ablowitz, M.J., Kaup, D.J., Newell, A.C., and Segur, H. Stud. Appl. Math. 53, 249 (1974).

[53] Hille, E.: Ordinary differential equations in the complex domain, Wiley-Interscience, NY 1976.

[54] Birkhoff, G.D.: Trans. AMS 436 (1909) and 199 (1910).

[55] Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions, NY, Dover, 1972; Whittaker, E.T., Watson, G.N.: A course in modern analysis, Cambridge Press, 1963.

[56] Kirmura, H., Okamoto, K.: J. Math. Pures Appl., 63, No. 1, 129 (1984); Okamoto, K.: Proc. Jap. Acad., Ser. A, 56, No. 6, 264 (1980).

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